

MONOTONICITY OF THE JUMP SET AND JUMP AMPLITUDES IN ONE-DIMENSIONAL TV DENOISING

RICCARDO CRISTOFERI, RITA FERREIRA, IRENE FONSECA, AND JOSÉ A. IGLESIAS

ABSTRACT. We revisit the classical problem of denoising a one-dimensional scalar-valued function by minimizing the sum of an L^2 fidelity term and the total variation, scaled by a regularization parameter. This study focuses on proving that the jump set of solutions, corresponding to discontinuities or edges, as well as the amplitude of the jumps are nonincreasing as the regularization parameter increases. Our results apply to input functions in L^∞ with left and right approximate limits everywhere, extending beyond the traditional setting of functions of bounded variation. The proof leverages competitor constructions and convexity properties of the taut string problem, a well-known equivalent formulation of the TV model. Such a monotonicity property reflects that the extent to which geometric and topological features of the original signal are preserved is consistent with the amount of smoothing desired when formulating the denoising method.

1. INTRODUCTION

The Total Variation (TV) model is a cornerstone of signal and image denoising due to its ability to reduce noise while preserving sharp edges. By minimizing a balance between the total variation and a fidelity term, typically measured in terms of an L^2 norm, the TV model effectively smooths flat regions without blurring important images' features, such as object boundaries, which makes it particularly suitable for applications where maintaining sharp geometric and topological characteristics is critical.

Since proposed by Rudin, Osher, and Fatemi [30], the TV model, also known as the ROF model, has inspired extensive research, including theoretical studies on the regularization properties [1, 10, 23], efficient numerical algorithms [9, 12], and extensions to anisotropic [20, 27] and higher-order formulations [5]. These developments underscore the versatility and impact of the TV model in both theory and practice.

A fundamental question associated with the TV and other denoising models is the study of the structure of their solutions' *jump set*, i.e., the points where discontinuities occur corresponding to edges or abrupt changes in signals. This study is key in the theoretical understanding of the regularization's role in signal processing and image analysis, aligning closely with practical applications in signal processing, where preserving or controlling edge structures is paramount.

Considerable attention has been given to the one-dimensional case, where the so-called taut string formulation offers a powerful lens to study solutions to the TV model. This is based on the equivalence between the TV and the taut string problems (see Problems 1.1 and 2.1 below). The latter consists in the minimization of the length of the graph of a function with fixed boundary conditions and bilateral inequality constraints given by shifted versions of an input function. This equivalence is by now well known, having been recognized in the discrete setting since [16], and used in the discretization of signal processing problems [22, 13] as well as in the continuum setting [19], with generalizations to more general constraints in [21] and weighted total variation in [4].

In this work, we further explore the equivalence between the two aforementioned problems to prove monotonicity properties of the jump set of the solutions to the TV model in the one-dimensional setting, which ensure that the locations of discontinuities (edges) in the denoised image align with the original image's structure. This key property helps maintaining the geometric and topological integrity of the image features, leading to more accurate and visually pleasing denoising results.

Our purpose is twofold. First, we address the question of how increasing the regularization parameter, α (see (1.1) below), simplifies the solution by progressively eliminating jumps and reducing their amplitude, thereby sharpening the correspondence between input signal features and regularized solutions. Second,

2020 *Mathematics Subject Classification.* 26A45, 26A46, 94A12, 68U10.

Key words and phrases. total variation, denoising, jump set, jump amplitude, regularization parameter, taut string method.

we seek to generalize results beyond bounded variation spaces, as explored in [14], by considering inputs that satisfy weaker conditions, such as the existence of left and right approximate limits everywhere.

The monotonicity of the jump set with respect to α ensures the hierarchical structure of solutions, wherein solutions for smaller α contain all the jump points present in solutions for larger α . This relationship provides a mathematical foundation for interpreting the sparsification effects of regularization.

Precisely, we address here the study of monotonicity properties of the jump set and of the amplitude of the jump of the solutions to the following one-dimensional signal denoising given by a ROF-type minimization:

Problem 1.1 (ROF). *Let $f \in L^2(0, 1)$ and $\alpha \in (0, +\infty)$, and consider the functional $\mathcal{I}_\alpha: BV(0, 1) \rightarrow [0, +\infty)$ defined, for $u \in BV(0, 1)$, by*

$$\mathcal{I}_\alpha[u] := \frac{1}{2} \|u - f\|_{L^2(0,1)}^2 + \alpha TV(u, (0, 1)). \quad (1.1)$$

Find $u_\alpha \in BV(0, 1)$ such that

$$\mathcal{I}_\alpha[u_\alpha] = \min_{u \in BV(0,1)} \mathcal{I}_\alpha[u]. \quad (1.2)$$

By using basic properties of $BV(0, 1)$ and an application of the direct method of the calculus of variations, this problem can easily be seen to have minimizers. Moreover, in (1.1) the first term is strictly convex while the second is also convex, so the minimizer u_α is unique. Functions in $BV(0, 1)$ need not be continuous, and in case the input function f is discontinuous, the minimizers are typically discontinuous as well. Our main result is the monotonicity with respect to α of the jump set of the u_α in the one-dimensional setting.

Theorem 1.2. *Let $f \in L^\infty(0, 1)$ be such that its left and right approximate limits*

$$f^-(x) := \lim_{y \rightarrow x^-} \int_y^x f(t) dt \quad \text{and} \quad f^+(x) := \lim_{y \rightarrow x^+} \int_x^y f(t) dt \quad (1.3)$$

exist at every point $x \in (0, 1)$. Let $\alpha_1, \alpha_2 \in (0, +\infty)$, and denote by $u_{\alpha_1}, u_{\alpha_2} \in BV(0, 1)$ the corresponding solutions to Problem 1.1. If $\alpha_1 \leq \alpha_2$, then

$$J_{u_{\alpha_2}} \subseteq J_{u_{\alpha_1}} \subseteq J_f.$$

Moreover, we have for each $x \in J_{u_{\alpha_2}}$ that

$$|u_{\alpha_1}^l(x) - u_{\alpha_1}^r(x)| \geq |u_{\alpha_2}^l(x) - u_{\alpha_2}^r(x)|,$$

where the left and right limits are defined in (1.6).

It was pointed out in [25] that, under the assumption $f \in L^\infty(0, 1) \cap BV(0, 1)$, the inclusion $J_{u_{\alpha_2}} \subseteq J_{u_{\alpha_1}}$ follows by combining the monotonicity of the jump set with respect to time for the TV flow of [8, Thm. 2] with the equivalence of varying the ROF parameter and advancing time in TV flow in the one-dimensional case proved in [6, Prop. 4.2]. Under such equivalence, one can also obtain the monotonicity of the jump amplitudes using the ‘local estimate’ $|Du| < |Df|$ of [18] or the estimates with respect to the jump amplitudes of f contained in [11, Thm. 3.1]. In contrast to that approach, the proof we present in this work applies to a larger class of input functions f (see Remark 1.4 below) and uses only variational arguments mainly based on the construction of competitors. As such, it is plausible that an approach such as ours may pave the way to a proof in more general settings, like the analog of Problem 1.1 for functions taking values in $\mathbb{R}^m$ with $m > 1$.

Remark 1.3. Monotonicity of the jump set of minimizers and their inclusion in the jump set of the input function is consistent with the limits $\alpha \rightarrow 0$ and $\alpha \rightarrow +\infty$. In the first case, assuming $f \in BV \cap L^2$ and using $u = f$ in (1.2) one readily obtains $u_\alpha \rightarrow f$ strongly in L^2 . For large α , one may prove by using convexity of (1.1) and a Poincaré inequality (see [24, Prop. 2.5.7]) that there is some α_f depending on f such that for $\alpha \geq \alpha_f$ the minimizers u_α are constant and equal to the average of f on $(0, 1)$. Both of these observations hold true also for the higher-dimensional case and beyond scalar-valued functions.

Remark 1.4. We observe that the space of functions satisfying (1.3) is strictly contains the space of functions of bounded variation. First of all, we note that, if $f \in BV$, then the above assumptions are satisfied. Indeed, fix $x_0 \in (0, 1)$ and consider the function $F : (0, 1) \rightarrow \mathbb{R}$ defined as

$$F(x) := \int_{x_0}^x f(t) dt.$$

Then, since $f \in L^\infty(0, 1)$, we get that F is Lipschitz, and therefore of bounded variation. Thus, the validity of (1.3) follows from [28, Thm 2.17].

We now provide two explicit examples of a function that satisfies assumption (1.3), but is not of bounded variation. The first example is the function $f : (0, 1) \rightarrow \mathbb{R}$ given by $f(x) := \sin(1/x)$. The function f is not of bounded variation, but it is analytic, and thus satisfies (1.3).

For the reader who is dissatisfied by the previous example since there is degeneracy at the boundary of the domain of definition, we present another more involved example. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 1 & \text{if } x \in [n^{-1}, n^{-1} + 2^{-n}], \text{ for some } n \geq 1, \\ 0 & \text{else.} \end{cases}$$

This function does not have bounded pointwise variation (and thus, it is not of bounded variation). Indeed, it has countably many jumps of size one. Nevertheless, it satisfies assumption (1.3). The only point where we need to check that the required condition is satisfied is $x_0 = 0$. Note that this point is in the interior of the domain of definition of the function f . We claim that

$$\lim_{x \rightarrow x_0^-} \int_x^{x_0} f(x) dt = \lim_{x \rightarrow x_0^+} \int_{x_0}^x f(x) dt = 0.$$

We only verify that the first limit holds, since the latter follows by using a similar argument. Let $x > 0$, and let $n \in \mathbb{N}$ be such that $x \in [n^{-1}, (n-1)^{-1}]$. Then,

$$\frac{1}{x} \int_0^x f(t) dt \leq \frac{1}{n} \int_0^{n^{-1}} f(t) dt = \frac{1}{n} \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{2n}{2^{n-1}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Using this example, it is possible to construct a function with countably many points that behave like the origin for the above function.

1.1. Strategy of the proof. The rest of the paper is dedicated to the proof of Theorem 1.2, building on intermediate results that may be of independent interest. Our arguments are hinged on the equivalence between the ROF Problem 1.1 and the taut string Problem 2.1 below, stating that the input and solution of Problem 1.1 are the derivatives of respectively the input and solution of Problem 2.1. As already mentioned above, the taut string problem consists in the minimization of the length of the graph of a function with fixed boundary conditions and bilateral inequality constraints given by shifted versions of a primitive of the input function f .

Section 2 is dedicated to basic results about the taut string problem. We start by recalling the equivalence in Theorem 2.2, along with the likewise known ‘universal minimality’ property of the taut string problem in Theorem 2.3, stating that the length integrand can be replaced by any other strictly convex function without changing the unique minimizer. Then, we define the contact sets consisting of points of the domain in which either of the constraints of the taut string problem is active, with Lemma 2.7 being dedicated to first consequences of this definition. In this lemma, we recall the (also known) convexity and concavity properties of the solutions in the complement of the lower and upper contact sets respectively, and prove a straightforward bound on the amount of times the solution can switch between the two contact sets. Finally, we prove in Proposition 2.8 that solutions of the taut string problem are stable in L^∞ norm with respect to the regularization parameter, combining the universal minimality property with a Γ -convergence argument.

Section 3 tackles the proof of Theorem 1.2, relying on properties from the previous section. The first main step is Theorem 3.2 in which precise optimality conditions at the contact sets are proved by a competitor construction which exploits the quantitative strict convexity of the graph length integrand. These state, roughly speaking, that the taut string minimization decreases the absolute slope in the minimizers with respect to the inputs. Taking advantage of these optimality conditions, the convexity/concavity

properties of Lemma 2.7 and the L^∞ stability of Proposition 2.8, we prove in Theorem 3.3 that the contact sets are nonincreasing with respect to the regularization parameter. Independently of that result, in Proposition 3.4 we obtain, directly from the optimality conditions, that the jump set of the solution of the ROF problem is contained in the intersection of the jump set of its input and the contact sets of the corresponding taut string problem. This last result is then used in Proposition 3.7 to conclude that monotonicity of the contact sets implies monotonicity of the jump sets. Finally, in Proposition 3.8, we show how the strategy we used to prove the monotonicity of the jump set can be used also to assert the monotonicity of the amplitude of the jump set, thereby completing the proof of Theorem 1.2.

1.2. Notation and preliminaries. The regularization term in (1.1) is the total variation, defined as

$$TV(u, (0, 1)) := \sup \left\{ \int_0^1 u(x) \psi'(x) dx : \psi \in C_c^1(0, 1), |\psi(x)| \leq 1 \text{ for all } x \in (0, 1) \right\},$$

with the space of functions of bounded variation being given by

$$BV(0, 1) := \{u \in L^1(0, 1) : TV(u, (0, 1)) < +\infty\} \text{ with } \|u\|_{BV(0, 1)} := \|u\|_{L^1(0, 1)} + TV(u, (0, 1)).$$

Equivalently, $BV(0, 1)$ can be seen as the subset of $u \in L^1(0, 1)$ whose distributional derivative Du is a finite signed Radon measure, that is,

$$\int_0^1 \psi(x) dDu(x) = - \int_0^1 u(x) \psi'(x) dx \text{ for all } \psi \in C_c^\infty(0, 1). \quad (1.4)$$

We will occasionally work with the usual Sobolev spaces $W^{k,p}(0, 1)$ of functions $u \in L^p(0, 1)$ whose distributional derivative Du (see (1.4)) satisfies $Du \in L^p(0, 1)$. Since these are all defined by integration, strictly speaking they are spaces of equivalence classes of functions, up to equality a.e. with respect to the Lebesgue measure $\mathcal{L}^1$ on $(0, 1)$. As we will be dealing with pointwise properties of these functions, we need to pick specific representatives so that equality between functions is understood to be everywhere in $(0, 1)$. For $u \in BV(0, 1)$, we will work with so-called good representatives $\tilde{u} : (0, 1) \rightarrow \mathbb{R}$ satisfying

$$TV(u, (0, 1)) = \sup \left\{ \sum_{i=1}^{n-1} |\tilde{u}(x_{i+1}) - \tilde{u}(x_i)| : n \geq 2, 0 < x_1 < \dots < x_n < 1 \right\}, \quad (1.5)$$

where the right-hand side is known as the pointwise variation of $\tilde{u}$. This property implies [2, Thm. 3.28] that the left and right limits

$$\lim_{y \rightarrow x^-} \tilde{u}(y) \text{ and } \lim_{y \rightarrow x^+} \tilde{u}(y) \text{ exist for all } x \in (0, 1),$$

and that these are equal for all x such that $Du(\{x\}) = 0$. Good representatives are not uniquely defined by the property (1.5); some specific choices are the left-continuous, right-continuous, and precise representatives defined for $x \in (0, 1)$ as

$$u^\ell(x) := u(0) + Du((0, x)), \quad u^r(x) := u(0) + Du((0, x]), \quad (1.6)$$

and

$$u^*(x) := \lim_{\delta \rightarrow 0^+} \int_{x-\delta}^{x+\delta} u(t) dt = \frac{u^\ell(x) + u^r(x)}{2},$$

respectively. Note that the limit in the definition of $u^*(x)$ is known as the approximate limit of u at x , and $u(0)$ is to be understood in the sense of traces, satisfying in particular that [17, Thm. 5.7]

$$\lim_{\delta \rightarrow 0^+} \int_0^\delta |u(t) - u(0)| dt = 0.$$

In the following, we always identify $u \in BV(0, 1)$ or $u \in W^{k,p}(0, 1)$ with their precise representative u^* .

We will also work with the space $AC(0, 1)$ of absolutely continuous functions, that is, $u \in AC(0, 1)$ whenever for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\sum_{i=1}^n |u(b_i) - u(a_i)| < \varepsilon \text{ for all pairwise disjoint } (a_i, b_i) \subset (0, 1), i = 1, \dots, n, \text{ with } \sum_{i=1}^n (b_i - a_i) < \delta.$$

Functions in $AC(0, 1)$ are differentiable $\mathcal{L}^1$ -a.e. [28, Prop. 3.9], and for them the identification with precise representatives implies $AC(0, 1) = W^{1,1}(0, 1)$ and $Du(x) = u'(x)$ for $\mathcal{L}^1$ -a.e. $x \in (0, 1)$. Moreover, under the same identification, we also have

$$W^{1,\infty}(0, 1) = \left\{ u : (0, 1) \rightarrow \mathbb{R} : \text{Lip}(u) := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|} < +\infty \right\}, \text{ and } \text{Lip}(u) = \|Du\|_{L^\infty(0,1)}.$$

Finally, for any locally integrable function $u \in L^1_{\text{loc}}(0, 1)$, one can define the jump set J_u as the set of $x \in (0, 1)$ for which the left and right approximate limits

$$u^-(x) := \lim_{y \rightarrow x^-} \int_y^x u(t) dt \quad \text{and} \quad u^+(x) := \lim_{y \rightarrow x^+} \int_x^y u(t) dt$$

both exist but are different. We remark that here and in the sequel, we will use this definition for u^- and u^+ , which differs from the common convention for BV functions in more than one dimension, in which these are respectively used to denote the lower and higher value at either side of a jump point. If additionally $u \in BV(0, 1)$, then (the precise representative of) u is also differentiable $\mathcal{L}^1$ -a.e., and we have the decomposition

$$Du = (u')\mathcal{L}^1 + [\max(u^-, u^+) - \min(u^-, u^+)]\mathcal{H}^0 \llcorner J_u + D^c u,$$

where $\mathcal{H}^0$ is the counting measure and the Cantor part $D^c u$ is singular with respect to the other two terms and vanishes on all countable subsets of $(0, 1)$.

2. THE TAUT STRING PROBLEM

Problem 2.1 (Taut String). Let $f \in L^2(0, 1)$ and $\alpha \in (0, +\infty)$. Define

$$F(t) := \int_0^t f(x) dx \quad \text{for } t \in [0, 1]. \quad (2.1)$$

Set

$$\mathcal{A}_\alpha := \left\{ U \in AC(0, 1) : \sup_{x \in (0, 1)} |U(x) - F(x)| \leq \alpha, U(0) = F(0) = 0, U(1) = F(1) \right\},$$

and consider the functional $\mathcal{J} : AC(0, 1) \rightarrow [0, +\infty)$ defined, for $U \in AC(0, 1)$, as

$$\mathcal{J}[U] := \int_0^1 \sqrt{1 + [U'(x)]^2} dx.$$

Find $U_\alpha \in \mathcal{A}_\alpha$ such that

$$\mathcal{J}[U_\alpha] = \min_{U \in \mathcal{A}_\alpha} \mathcal{J}[U]. \quad (2.2)$$

By [19, Lemma 2], there is a unique solution U_α to Problem 2.1 (see Figure 1 for a numerical example). We now state the equivalence between the ROF Problem 1.1 and the taut string Problem 2.1, which is a special case of a more general *universal minimality* of the solution to the ROF Problem 1.1, as addressed in Theorem 2.3 and Remark 2.4.

Theorem 2.2 (cf. [19, Proposition 2]). Assume that $f \in L^\infty(0, 1)$, and fix $\alpha \in (0, +\infty)$. Let $u_\alpha \in BV(0, 1)$ be the unique solution to Problem 1.1 and $U_\alpha \in \mathcal{A}_\alpha$ the unique solution to Problem 2.1. Then, $U'_\alpha \in BV(0, 1)$ and

$$u_\alpha = U'_\alpha,$$

where the equation holds for the good representatives.

As mentioned above, it turns out that minimizers of the ROF Problem (1.1) are also minimizers of a large class of variational problems. This surprising fact is proved in [31, Theorems 4.35 and 4.46]. Due to its importance for our strategy (and its interest), we prove it in here by detailing the proof in the aforementioned reference.

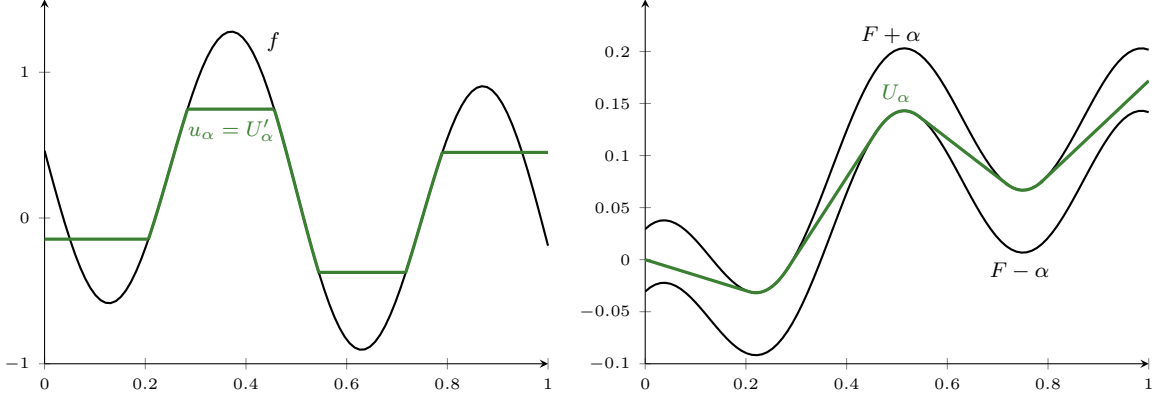


FIGURE 1. ROF solution u_α , constraints $F - \alpha$, $F + \alpha$ and taut string solution U_α with $f(x) = \sin(4\pi(x + 1/4)) + \cos(\pi/2(x + 1/4))/2$ and $\alpha = 0.03$.

Theorem 2.3 (Universal minimality condition). Fix $\alpha \in (0, +\infty)$, and let $U_\alpha \in \mathcal{A}_\alpha$ be the unique solution to Problem 2.1. Then, U_α is also the unique function satisfying

$$U_\alpha \in \operatorname{argmin} \left\{ \int_0^1 c(v') \, dx : v \in W^{1,2}(0,1), \|v - F\|_\infty \leq \alpha, v(0) = F(0) = 0, v(1) = F(1) \right\}, \quad (2.3)$$

for any strictly convex function $c : \mathbb{R} \rightarrow \mathbb{R}$.

Proof. By [31, Theorem 4.46], we know that U'_α is the unique function satisfying

$$\int_0^1 c(U'_\alpha) \, dx = \min \left\{ \int_0^1 c(w) \, dx : w \in L^2(0,1), \|w - f\|_* \leq \alpha \right\}, \quad (2.4)$$

where

$$\|\varphi\|_* := \sup \left\{ \int_0^1 \varphi \phi \, dx : \phi \in L^2(0,1), TV(\phi, (0,1)) \leq 1 \right\}. \quad (2.5)$$

By setting $\hat{w}(t) := \int_0^t w(s) \, ds$, we can re-write (2.4) as

$$\int_0^1 c(U'_\alpha) \, dx = \min \left\{ \int_0^1 c(\hat{w}') \, dx : \hat{w} \in W^{1,2}(0,1), \hat{w}(0) = 0, \|\hat{w}' - f\|_* \leq \alpha \right\}.$$

Thus, to conclude the proof, it suffices to show that that

$$\begin{aligned} \{v \in W^{1,2}(0,1) : \|v - F\|_\infty \leq \alpha, v(0) = F(0), v(1) = F(1)\} \\ = \{\hat{w} \in W^{1,2}(0,1) : \hat{w}(0) = 0, \|\hat{w}' - f\|_* \leq \alpha\}. \end{aligned} \quad (2.6)$$

Assume that $v \in W^{1,2}(0,1)$ is such that $\|v - F\|_\infty \leq \alpha$, $v(0) = F(0)$, and $v(1) = F(1)$. Then, the condition $F' = f$ $\mathcal{L}^1$ -a.e. in $(0,1)$ yields

$$\begin{aligned} \|v' - f\|_* &= \sup \left\{ \int_0^1 (v - F)' \phi \, dx : \phi \in L^2(0,1), TV(\phi, (0,1)) \leq 1 \right\} \\ &= \sup \left\{ \int_0^1 (F - v) \, dD\phi : \phi \in L^2(0,1), TV(\phi, (0,1)) \leq 1 \right\} \\ &\leq \sup \left\{ \|v - F\|_\infty TV(\phi, (0,1)) : \phi \in L^2(0,1), TV(\phi, (0,1)) \leq 1 \right\} \\ &\leq \alpha. \end{aligned}$$

Hence, recalling that $F(0) = 0$, the set on the left-hand side of (2.6) is contained in the set on the right-hand side of (2.6).

Conversely, assume that $\hat{w} \in W^{1,2}(0,1)$ is such that $\hat{w}(0) = 0$ and $\|\hat{w}' - f\|_* \leq \alpha$. Then, setting $\phi := \lambda \in \mathbb{R}$ in (2.5) we get that

$$\alpha \geq \lambda \int_0^1 (\hat{w} - F)' \, dx = \lambda(\hat{w}(1) - F(1))$$

for all $\lambda \in \mathbb{R}$. Thus, $\hat{w}(1) = F(1)$.

Now, for each $x_0 \in (0, 1)$, define the test function

$$\phi_{x_0}(x) := \begin{cases} 1 & \text{if } 0 < x \leq x_0, \\ 0 & \text{if } x_0 < x < 1. \end{cases}$$

With $\phi := \pm \phi_{x_0}$ as test functions in (2.5), yields

$$\alpha \geq \pm \int_0^1 (\hat{w} - F)' \phi_{x_0} \, dx = \pm \int (F - \hat{w}) \, dD\phi_{x_0} = \pm (F - \hat{w})(x_0),$$

where we used the identities $\hat{w}(0) = 0 = F(0)$ and $\hat{w}(1) = F(1)$. Because x_0 was arbitrarily taken, the preceding estimate shows that $\|\hat{w} - F\|_\infty \leq \alpha$, and this concludes the proof of (2.6). $\square$

Remark 2.4. This universal minimality criterion allows, in particular, to relate Problem 2.1 with a canonical choice of Fenchel (pre)dual of Problem 1.1, which results in precisely (2.3) with the integrand $c(t) := t^2/2$. This fact is well-known, for a proof we refer the reader to [29] in the context of continuum taut string methods and [4] allowing for weights in the total variation and fidelity term.

We now introduce some sets associated with the taut string problem that will be used later in the proof of the main result.

Definition 2.5. Given $\alpha \in (0, +\infty)$, let U_α be the unique solution to the taut string problem, Problem 2.1. Define the upper and lower contact sets as

$$\begin{aligned} C_\alpha^+ &:= \{x \in (0, 1) : U_\alpha(x) = F(x) + \alpha\}, \\ C_\alpha^- &:= \{x \in (0, 1) : U_\alpha(x) = F(x) - \alpha\}, \end{aligned} \tag{2.7}$$

respectively.

Remark 2.6. We observe that the sets $(0, 1) \setminus (C_\alpha^- \cup C_\alpha^+)$, C_α^+ , and C_α^- are mutually disjoint, and their union equals the interval $(0, 1)$ because $U_\alpha \in \mathcal{A}_\alpha$.

It is possible to obtain convexity and concavity properties of U_α in $(0, 1) \setminus C_\alpha^+$ and $(0, 1) \setminus C_\alpha^-$ respectively, as already done in [21] and recalled in the next lemma. Moreover, we also give a basic bound (see [3, Lemma 9] for related result in the context of minimization of the number of modes via taut string minimization) on how often the solution U_α can switch from C_α^- to C_α^+ .

Lemma 2.7. Fix $\alpha \in (0, +\infty)$, let $U_\alpha \in \mathcal{A}_\alpha$ be the unique solution to Problem 2.1. Then,

- (1) U_α is convex on each connected component of $(0, 1) \setminus C_\alpha^-$;
- (2) U_α is concave on each connected component of $(0, 1) \setminus C_\alpha^+$;

In particular, U_α is affine on each connected component of $(0, 1) \setminus (C_\alpha^- \cup C_\alpha^+)$. Moreover,

- (3) For each $x^- \in C_\alpha^-$ and $x^+ \in C_\alpha^+$, we have

$$|x^- - x^+| > \frac{2\alpha}{\text{Lip}(F)}, \tag{2.8}$$

where $\text{Lip}(F)$ is the Lipschitz constant of F . Consequently,

$$U_\alpha \text{ is convex on } \left[x^+ - \frac{2\alpha}{\text{Lip}(F)}, x^+ + \frac{2\alpha}{\text{Lip}(F)} \right], \text{ and concave on } \left[x^- - \frac{2\alpha}{\text{Lip}(F)}, x^- + \frac{2\alpha}{\text{Lip}(F)} \right].$$

Proof. Step 1. We prove (1) and (2). Their proof can be found in a slightly more general setting in [21, Theorem 3.1, Lemma 5.4], but we report their argument here for completeness. For that, let $x_0 \in (0, 1) \setminus C_\alpha^-$, and denote

$$t := U_\alpha(x_0) - F(x_0) + \alpha > 0.$$

By continuity of U_α and F , there is $\delta > 0$ so that

$$|U_\alpha(x) - U_\alpha(x_0)| < \frac{t}{2}, \quad |F(x) - F(x_0)| < \frac{t}{2} \quad \text{for all } x \in (x_0 - \delta, x_0 + \delta) \subset [0, 1]. \quad (2.9)$$

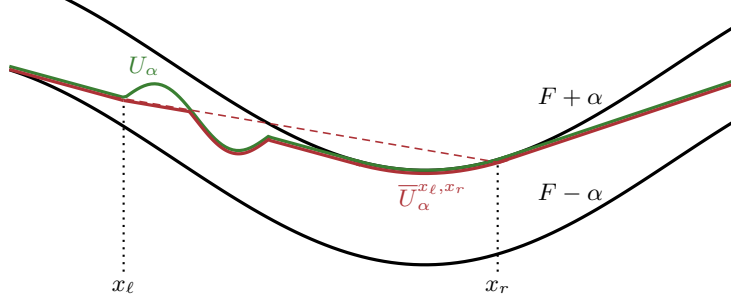


FIGURE 2. Competitor construction from [21, Lem. 5.4]. If x_ℓ, x_r are in the same connected component of $(0, 1) \setminus C_\alpha^-$ and close enough to each other, $\bar{U}_\alpha^{x_\ell, x_r}$ is admissible for the taut string problem and minimality of U_α against $\bar{U}_\alpha^{x_\ell, x_r}$ certifies the convexity inequality $U_\alpha((1 - \lambda)x_\ell + \lambda x_r) \leq (1 - \lambda)U_\alpha(x_\ell) + \lambda U_\alpha(x_r)$ for all $\lambda \in (0, 1)$.

We then fix two points $x_\ell, x_r \in \mathbb{R}$ with $x_0 - \delta < x_\ell < x_r < x_0 + \delta$ and define $\bar{U}_\alpha^{x_\ell, x_r} : [0, 1] \rightarrow \mathbb{R}$ as (see Figure 2)

$$\bar{U}_\alpha^{x_\ell, x_r}(x) := \begin{cases} U_\alpha(x) & \text{if } x \in [0, x_\ell] \cup [x_r, 1], \\ \min \{U_\alpha(x), (1 - \lambda)U_\alpha(x_\ell) + \lambda U_\alpha(x_r)\} & \text{if } x = (1 - \lambda)x_\ell + \lambda x_r \text{ with } \lambda \in (0, 1). \end{cases}$$

We claim that U_α is convex on $(x_0 - \delta, x_0 + \delta)$. We then conclude that U_α is convex on each connected component of $(0, 1) \setminus C_\alpha^-$ as follows. Let a, b be in a connected component of $(0, 1) \setminus C_\alpha^-$. Then, it is possible to find finitely many points $a = y^1 < \dots < y^k = b$ such that $(x_\ell^i, x_r^i) \cap (x_\ell^{i+1}, x_r^{i+1}) \neq \emptyset$, where x_ℓ^i, x_r^i are the points constructed above relative to y^i . For the sake of the argument, we assume $k = 2$. The general case can be treated similarly. Fix $\lambda \in (0, 1)$, and let $c := \lambda a + (1 - \lambda)b$. Then, it is possible to write

$$c = \mu a + (1 - \mu)x_r^1, \quad x_r^1 = \omega x_\ell^2 + (1 - \omega)b, \quad x_\ell^2 = \xi a + (1 - \xi)x_r^1,$$

for some $\mu, \omega, \xi \in [0, 1]$. This yields that

$$x_r^1 = \frac{\omega\xi}{1 - \omega(1 - \xi)}a + \frac{1 - \omega}{1 - \omega(1 - \xi)}b.$$

Therefore, we get

$$\begin{aligned} f(c) &\leq \mu f(a) + (1 - \mu)f(x_r^1) \\ &\leq \mu f(a) + (1 - \mu)\frac{\omega\xi}{1 - \omega(1 - \xi)}f(a) + (1 - \mu)\frac{1 - \omega}{1 - \omega(1 - \xi)}f(b). \end{aligned}$$

Noting that

$$c = [\mu + (1 - \mu)\frac{\omega\xi}{1 - \omega(1 - \xi)}]a + (1 - \mu)\frac{1 - \omega}{1 - \omega(1 - \xi)}b,$$

we obtain the desired conclusion.

We now prove the claim. Note that convexity of U_α on $(x_0 - \delta, x_0 + \delta)$ is equivalent to

$$U_\alpha(x) \leq \bar{U}_\alpha^{x_\ell, x_r}(x)$$

for all $x \in (x_\ell, x_r)$ and all $x_\ell, x_r \in (x_0 - \delta, x_0 + \delta)$. To prove this inequality we assume, on the contrary, that there is some $y \in (x_\ell, x_r)$ with $U_\alpha(y) > \bar{U}_\alpha^{x_\ell, x_r}(y)$. Then, again by continuity, there would be $\eta > 0$ for which

$$U_\alpha(x) > \bar{U}_\alpha^{x_\ell, x_r}(x)$$

for all $x \in (y - \eta, y + \eta)$, and

$$U_\alpha(y - \eta) = \bar{U}_\alpha^{x_\ell, x_r}(y - \eta), \quad U_\alpha(y + \eta) = \bar{U}_\alpha^{x_\ell, x_r}(y + \eta).$$

We now claim that

$$\mathcal{J}[\overline{U}_\alpha^{x_\ell, x_r}] < \mathcal{J}[U_\alpha]. \quad (2.10)$$

Indeed, by definition, we have that

$$\int_{[0, x_\ell] \cup [x_r, 1]} \sqrt{1 + [(\overline{U}_\alpha^{x_\ell, x_r})']^2} dx = \int_{[0, x_\ell] \cup [x_r, 1]} \sqrt{1 + (U'_\alpha(x))^2} dx.$$

Moreover, we can write

$$\begin{aligned} [x_\ell, x_r] &= \{x = (1 - \lambda)x_\ell + \lambda x_r \in [x_\ell, x_r] : U_\alpha(x) \leq (1 - \lambda)U_\alpha(x_\ell) + \lambda U_\alpha(x_r)\} \\ &\cup \{x = (1 - \lambda)x_\ell + \lambda x_r \in [x_\ell, x_r] : U_\alpha(x) > (1 - \lambda)U_\alpha(x_\ell) + \lambda U_\alpha(x_r)\} =: B \cup A, \end{aligned}$$

namely, as a disjoint union of two measurable sets. The set A is an open set, therefore with at most countably many connected components. Thus, there exists a set of indexes I , which is at most countable, such that

$$A = \bigcup_{i \in I} (a_i, b_i),$$

for some $a_i < b_i$. Now, note that

$$(\overline{U}_\alpha^{x_\ell, x_r})'(x) = U'_\alpha(x)$$

for almost every $x \in B$. This gives

$$\int_B \sqrt{1 + [(\overline{U}_\alpha^{x_\ell, x_r})']^2} dx = \int_B \sqrt{1 + (U'_\alpha(x))^2} dx.$$

On the other hand, for each $i \in I$, we have that

$$\int_{(a_i, b_i)} \sqrt{1 + [(\overline{U}_\alpha^{x_\ell, x_r})']^2} dx < \int_{(a_i, b_i)} \sqrt{1 + (U'_\alpha(x))^2} dx,$$

since $\overline{U}_\alpha^{x_\ell, x_r}$ is affine on $[a_i, b_i]$, $\overline{U}_\alpha^{x_\ell, x_r}(a_i) = U_\alpha(a_i)$, and $\overline{U}_\alpha^{x_\ell, x_r}(b_i) = U_\alpha(b_i)$. Note that the strict inequality comes from the fact that $U_\alpha(x) > (1 - \lambda)U_\alpha(x_\ell) + \lambda U_\alpha(x_r)$ for all $x \in (a_i, b_i)$. This proves (2.10).

On the other hand, the function $\overline{U}_\alpha^{x_\ell, x_r}$ is admissible for (2.2). Indeed,

$$\overline{U}_\alpha^{x_\ell, x_r} \leq U_\alpha \leq F + \alpha,$$

and using (2.9) we get

$$\begin{aligned} &(1 - \lambda)U_\alpha(x_\ell) + \lambda U_\alpha(x_r) - F((1 - \lambda)x_\ell + \lambda x_r) + \alpha \\ &> (1 - \lambda) \left(U_\alpha(x_0) - \frac{t}{2} \right) + \lambda \left(U_\alpha(x_0) - \frac{t}{2} \right) - F(x_0) - \frac{t}{2} + \alpha \\ &= U_\alpha(x_0) - F(x_0) + \alpha - t \geq 0. \end{aligned}$$

Therefore, the minimality of U_α yields

$$\mathcal{J}[U_\alpha] \leq \mathcal{J}[\overline{U}_\alpha^{x_\ell, x_r}],$$

giving the desired contradiction.

Analogously, we obtain that U_α is concave on each connected component of $(0, 1) \setminus C_\alpha^+$.

Step 2. We prove (3). The estimate (2.8) follows immediately from the definition of $C_\alpha^\pm$ and the bound

$$\text{Lip}(U_\alpha) \leq \text{Lip}(F).$$

To prove the latter statement of the result, we notice that by Theorem 2.2, it is equivalent to $\|u_\alpha\|_\infty \leq \|f\|_\infty$, which in turn is a well known fact that follows by truncation. Indeed, if we consider $\hat{u}_\alpha := \min(u_\alpha, \|f\|_\infty)$, then we have

$$\|\hat{u}_\alpha - f\|_{L^2(0,1)} \leq \|u_\alpha - f\|_{L^2(0,1)} \quad \text{and} \quad TV(\hat{u}_\alpha, (0, 1)) \leq TV(u_\alpha, (0, 1)),$$

with both inequalities becoming strict if we had $\|\max(u_\alpha, 0)\|_\infty > \|\max(f, 0)\|_\infty$. An analogous argument applies for $\max(u_\alpha, -\|f\|_\infty)$, and combining both we get that $\|u_\alpha\|_\infty > \|f\|_\infty$ would contradict minimality of u_α for (1.2). $\square$

As a straightforward consequence of the above universal minimality, we obtain that the solutions of the taut string Problem 2.1 are continuous in the L^∞ topology with respect to α .

Proposition 2.8. *Let $\{\alpha_n\}_{n \in \mathbb{N}} \subset (0, +\infty)$ be such that $\alpha_n \rightarrow \alpha \in (0, +\infty)$. Then,*

$$U_{\alpha_n} \rightarrow U_\alpha \quad (2.11)$$

with respect to the uniform convergence.

Proof. In order to prove that U_{α_n} converges to U_α with respect to the uniform convergence, we argue as follows. For $t \geq 0$, define $\mathcal{F}_t : W^{1,2}(0, 1) \rightarrow \mathbb{R} \cup \{+\infty\}$ as

$$\mathcal{F}_t(v) := \begin{cases} \int_0^1 |v'|^2 dx & \text{if } \|v - F\|_\infty \leq t, \ v(0) = F(0) = 0, \ v(1) = F(1), \\ +\infty & \text{else.} \end{cases}$$

We claim that:

- (i) The sequence $\{\mathcal{F}_{\alpha_n}\}_{n \in \mathbb{N}}$ Γ -converges to $\mathcal{F}_\alpha$ as $n \rightarrow \infty$, with respect to the uniform convergence;
- (ii) If $\{w_n\}_{n \in \mathbb{N}} \subset W^{1,2}(0, 1)$ is such that

$$\sup_{n \in \mathbb{N}} \mathcal{F}_{\alpha_n}(w_n) < \infty,$$

then there exists a subsequence $\{w_{n_k}\}_{k \in \mathbb{N}}$ such that $w_{n_k} \rightarrow v$ uniformly, and $\mathcal{F}_\alpha(v) < \infty$;

- (iii) It holds that

$$\sup_{n \in \mathbb{N}} \mathcal{F}_{\alpha_n}(U_{\alpha_n}) < \infty.$$

This will allow us to conclude (2.11). Indeed, by using standard properties of Γ -convergence [15, Corollary 7.20] together with (iii) and (ii), we obtain that $U_{\alpha_n} \rightarrow v$ uniformly, where v is a minimizer of $\mathcal{F}_\alpha$. By strict convexity of the energy, together with Theorem 2.3, we know that U_α is the unique solution to the constrained minimization problem

$$\min \left\{ \int_0^1 |v'|^2 dx : v \in W^{1,2}(0, 1), \|v - F\|_\infty \leq \alpha_n, \ v(0) = F(0) = 0, \ v(1) = F(1) \right\}.$$

Thus, $U_{\alpha_n} \rightarrow U_\alpha$ uniformly, as desired.

We are thus left to prove (i), (ii), and (iii). We start with (i). We will show that

- (a) Liminf inequality: Let $\{v_n\}_{n \in \mathbb{N}}$ be such that $v_n \rightarrow v$ uniformly. Then,

$$\liminf_{n \rightarrow \infty} \mathcal{F}_{\alpha_n}(v_n) \geq \mathcal{F}_\alpha(v);$$

- (b) Limsup inequality: Let $v \in W^{1,2}(0, 1)$. Then, there exists $\{v_n\}_{n \in \mathbb{N}}$ such that $v_n \rightarrow v$ uniformly, and

$$\limsup_{n \rightarrow \infty} \mathcal{F}_{\alpha_n}(v_n) \leq \mathcal{F}_\alpha(v).$$

We first prove (a). Without loss of generality, we can assume that

$$\liminf_{n \rightarrow \infty} \mathcal{F}_{\alpha_n}(v_n) = \lim_{n \rightarrow \infty} \mathcal{F}_{\alpha_n}(v_n) < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \mathcal{F}_{\alpha_n}(v_n) < \infty.$$

This implies in particular that

$$v_n(0) = F(0), \quad v_n(1) = F(1), \quad \|v_n - F\|_\infty \leq \alpha_n,$$

for all $n \in \mathbb{N}$. By uniform convergence, by sending $n \rightarrow \infty$, we get that

$$v(0) = F(0), \quad v(1) = F(1), \quad \|v - F\|_\infty \leq \alpha. \quad (2.12)$$

To prove the desired inequality, we argue as follows. The sequence $\{v_n\}_{n \in \mathbb{N}}$ is bounded in $W^{1,2}(0, 1)$. Therefore, up to a not relabeled subsequence, it holds that $v_n \rightharpoonup u$ weakly in $W^{1,2}(0, 1)$. Since we already know that $v_n \rightarrow v$ uniformly, we get that $v_n \rightharpoonup v$ weakly in $W^{1,2}(0, 1)$. Thus, the liminf inequality follows from the lower semi-continuity of the norm with respect to weak convergence.

We now prove (b). Let $v \in W^{1,2}(0,1)$. Without loss of generality, we can assume that $\mathcal{F}_\alpha(v) < \infty$, otherwise there is nothing to prove. For $n \in \mathbb{N}$, define

$$v_n := \frac{\alpha_n}{\alpha}(v - F) + F.$$

Note that, for each $n \in \mathbb{N}$, $v_n \in L^\infty(0,1)$, because $W^{1,2}(0,1) \subset L^\infty(0,1)$. Moreover, using (2.12), it follows that

$$\|v_n - F\|_\infty = \frac{\alpha_n}{\alpha}\|v - F\|_\infty \leq \alpha_n, \quad v_n(1) = F(1), \quad v_n(0) = F(0).$$

Also,

$$\|v_n - v\|_\infty \leq \left| \frac{\alpha_n}{\alpha} - 1 \right| \|v - F\|_\infty$$

and, recalling that $F' = f$ $\mathcal{L}^1$ -a.e. in $(0,1)$,

$$\|v'_n\|_{L^2(0,1)} \leq \frac{\alpha_n}{\alpha}\|v'\|_{L^2(0,1)} + \left| \frac{\alpha_n}{\alpha} - 1 \right| \|f\|_{L^2(0,1)}.$$

Passing to the limit as $n \rightarrow \infty$ the two preceding estimates, the convergence $\alpha_n \rightarrow \alpha$ yields

$$\lim_{n \rightarrow \infty} \|v_n - v\|_\infty = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \int_0^1 |v'_n|^2 dx \leq \int_0^1 |v'|^2 dx,$$

which concludes the proof of (b).

We now prove (iii). By Theorem 2.2, we get that

$$\mathcal{F}_{\alpha_n}(U_{\alpha_n}) = \|U'_{\alpha_n}\|_{L^2(0,1)} = \|u_{\alpha_n}\|_{L^2(0,1)} \leq C \|u_{\alpha_n}\|_{BV(0,1)}.$$

The latter are uniformly bounded by definition of the ROF problem, since $\sup_n \alpha_n < \infty$, and this concludes the proof. $\square$

3. PROOF OF THE MONOTONICITY, THEOREM 1.2

We start by proving necessary optimality conditions at the contact sets in terms of left and right derivatives of F and of U , that will be used throughout our analysis.

Remark 3.1. Assume that $f \in L^\infty(0,1)$, fix $\alpha \in (0, +\infty)$, and let $U_\alpha \in \mathcal{A}_\alpha$ be the unique solution to Problem 2.1. Because $U_\alpha \in AC(0,1)$, we have for all $x_0 \in (0,1)$ that

$$\frac{U_\alpha(x) - U_\alpha(x_0)}{x - x_0} = \int_{x_0}^x U'_\alpha(t) dt.$$

Consequently, we conclude from the condition $U'_\alpha \in BV(0,1)$ by Theorem 2.2 that the limits

$$U'^+_\alpha(x_0) := \lim_{x \rightarrow x_0^+} \frac{U_\alpha(x) - U_\alpha(x_0)}{x - x_0} \quad \text{and} \quad U'^-_\alpha(x_0) := \lim_{x \rightarrow x_0^-} \frac{U_\alpha(x) - U_\alpha(x_0)}{x - x_0} \quad (3.1)$$

exist for all $x_0 \in (0,1)$, with $U'^+_\alpha(x_0) = U'^-_\alpha(x_0)$ for $\mathcal{L}^1$ -a.e. $x_0 \in (0,1)$.

Theorem 3.2. Let $f \in L^\infty(0,1)$ be such that the approximate left and right limits in (1.3) exist for all $x_0 \in (0,1)$. Fix $\alpha \in (0, +\infty)$, and let $U_\alpha \in \mathcal{A}_\alpha$ be the unique solution of Problem 2.1, and let $C^\pm_\alpha$ be the sets in (2.7). For $x_0 \in (0,1)$, let $U'^+_\alpha(x_0)$ and $U'^-_\alpha(x_0)$ be given by (3.1). Then, the following statements (illustrated in Figure 3) hold for any $x_0 \in (0,1)$:

(DL) If $x_0 \in C^-_\alpha$ and there exist $\delta > 0$ such that $(x_0 - \delta, x_0) \subset (0,1) \setminus (C^-_\alpha \cup C^+_\alpha)$, then

$$f^+(x_0) \leq U'^+_\alpha(x_0) \leq U'^-_\alpha(x_0) \leq f^-(x_0);$$

(DR) If $x_0 \in C^-_\alpha$ and there exists $\delta > 0$ such that $(x_0, x_0 + \delta) \subset (0,1) \setminus (C^-_\alpha \cup C^+_\alpha)$, then

$$f^+(x_0) \leq U'^+_\alpha(x_0) \leq U'^-_\alpha(x_0) \leq f^-(x_0);$$

(UL) If $x_0 \in C^+_\alpha$ and there exist $\delta > 0$ such that $(x_0 - \delta, x_0) \subset (0,1) \setminus (C^-_\alpha \cup C^+_\alpha)$, then

$$f^-(x_0) \leq U'^-_\alpha(x_0) \leq U'^+_\alpha(x_0) \leq f^+(x_0);$$

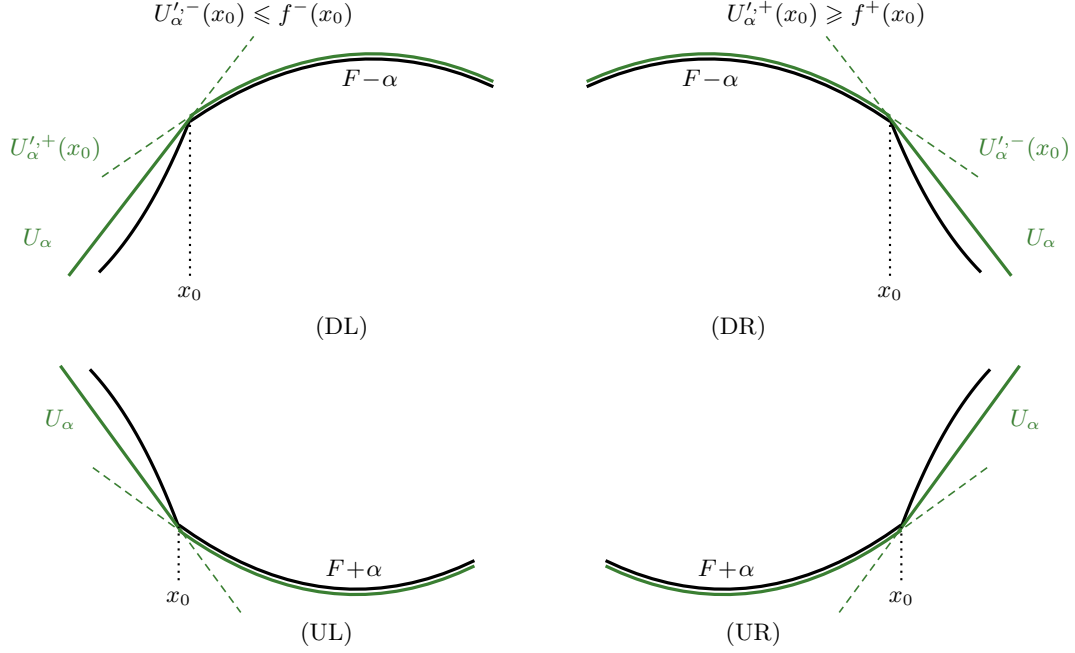


FIGURE 3. The different cases of the result of Theorem 3.2.

(UR) If $x_0 \in C_\alpha^-$ and there exists $\delta > 0$ such that $(x_0, x_0 + \delta) \subset (0, 1) \setminus (C_\alpha^- \cup C_\alpha^+)$, then

$$f^-(x_0) \leq U_\alpha'^-(x_0) \leq U_\alpha'^+(x_0) \leq f^+(x_0).$$

Proof. We treat each case separately.

Step 1. We address the (DL) case. We first prove that $U_\alpha'^-(x_0) \leq f^-(x_0)$. By definition of C_α^- and of $\mathcal{A}_\alpha$, we get that $U_\alpha(x_0) = F(x_0) - \alpha$ and $U_\alpha \geq F - \alpha$, respectively. Hence,

$$U_\alpha'^-(x_0) = \lim_{x \rightarrow x_0^-} \frac{U_\alpha(x) - U_\alpha(x_0)}{x - x_0} \leq \lim_{x \rightarrow x_0^-} \frac{F(x) - F(x_0)}{x - x_0} = f^-(x_0),$$

as desired.

The proof of $f^+(x_0) \leq U_\alpha'^+(x_0)$ follows by using a similar argument.

We now show that $U_\alpha'^-(x_0) \geq U_\alpha'^+(x_0)$. For the sake of notation, we write

$$\beta^- := U_\alpha'^-(x_0), \quad \beta^+ := U_\alpha'^+(x_0),$$

and, by contradiction, let us assume that $\beta^- < \beta^+$. We construct a competitor $\tilde{U}_\alpha$ by replacing part of U_α with a piecewise function, whose precise definition will depend on whether $\beta^+ = 0$ or not.

We first consider the $\beta^+ > 0$ case. Let $0 < \mu < 2\alpha/\text{Lip}(F)$ and $0 < \lambda < \delta$, with $\lambda \leq \mu$. We define $\tilde{U}_\alpha : [0, 1] \rightarrow \mathbb{R}$ as (see Figure 4)

$$\tilde{U}_\alpha(x) := \begin{cases} U_\alpha(x) & \text{if } x < x_0 - \lambda, \\ U_\alpha(x_0 - \lambda) + \frac{1}{2}(\beta^+ + \beta^-)(x - (x_0 - \lambda)) & \text{if } x_0 - \lambda \leq x < x_0 + \lambda, \\ U_\alpha(x_0 - \lambda) + (\beta^+ + \beta^-)\lambda & \text{if } x_0 + \lambda \leq x \leq x_0 + \mu, \\ U_\alpha(x) & \text{if } x > x_0 + \mu. \end{cases}$$

We claim that it is possible to choose λ and μ such that the following conditions hold:

- (i) $U_\alpha \in AC(0, 1)$;
- (ii) U_α is an admissible competitor for the minimization problem (2.2);
- (iii) $\mathcal{J}[\tilde{U}_\alpha] < \mathcal{J}[U_\alpha]$.

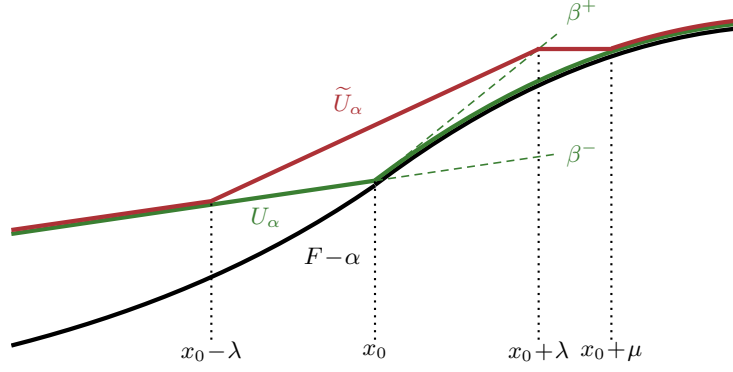


FIGURE 4. Construction of the competitor $\tilde{U}_\alpha$ for $\beta^+ > 0$, exploiting the inequality $\beta^- < \beta^+$ to obtain $\mathcal{J}[\tilde{U}_\alpha] < \mathcal{J}[U_\alpha]$, a contradiction.

This will provide the desired contradiction, since we constructed an admissible competitor whose energy is strictly lower than that of the minimizer.

To prove this energy inequality, we will use the following strict quantitative convexity of the function $t \mapsto \sqrt{1+t^2}$: There exist $\varepsilon_0 > 0$ and $s_0 > 0$ such that whenever $|\xi - \beta^+| < \varepsilon_0$ and $s < s_0$, it holds that

$$s + \sqrt{1 + \left(\frac{1}{2}\beta^- + \frac{1}{2}\beta^+\right)^2} < \frac{1}{2}\sqrt{1 + (\beta^-)^2} + \frac{1}{2}\sqrt{1 + \xi^2}. \quad (3.2)$$

Next, we observe that Lemma 2.7 and the assumptions on x_0 yield that that U_α is affine on $(x_0 - \delta, x_0]$ and concave on $[x_0, x_0 + 2\alpha/\text{Lip}(F))$. In particular, we have that

$$U_\alpha(y) = U_\alpha(x_0) + \beta^-(y - x_0) \quad \text{for all } y \in (x_0 - \delta, x_0], \quad (3.3)$$

$$U_\alpha(y) \leq U_\alpha(x_0) + \beta^+(y - x_0) \quad \text{for all } y \in [x_0, x_0 + 2\alpha/\text{Lip}(F)), \quad (3.4)$$

$$\beta_y := \frac{U_\alpha(y) - U_\alpha(x_0)}{y - x_0} \uparrow \beta^+ \quad \text{as } y \downarrow x_0^+, \quad \text{with } y < x_0 + 2\alpha/\text{Lip}(F). \quad (3.5)$$

On the other hand, using the continuity of U_α and F , together with the identity $U_\alpha(x_0) = F(x_0) - \alpha$ and the bound $\|U_\alpha - F\|_\infty \leq \alpha$, allows us to find $\bar{\delta} > 0$ such that

$$U_\alpha(x) \leq F(x) \quad \text{and} \quad |U_\alpha(x) - U_\alpha(x_0)| < \frac{\alpha}{2} \quad \text{whenever } |x - x_0| \leq \bar{\delta}. \quad (3.6)$$

Since $\beta^+ = U_{\alpha,-}'(x_0) > 0$ and U_α is concave on $[x_0, x_0 + 2\alpha/\text{Lip}(F))$, up to reducing $\bar{\delta}$, we can also ensure that

$$U_\alpha \text{ is increasing in } [x_0, x_0 + \bar{\delta}], \quad (3.7)$$

and we restrict $\bar{\delta}$ to satisfy

$$\bar{\delta} \leq \min \left\{ \bar{\delta}, \frac{\alpha}{2(|\beta^+ + \beta^-| + |\beta^+ - \beta^-| + \beta^+)}, \frac{2\alpha}{\text{Lip}(F)} \right\}. \quad (3.8)$$

Finally, recalling (3.5), we fix $\bar{y} \in (x_0, x_0 + \bar{\delta})$ such that

$$0 < \beta^+ - \beta_{\bar{y}} < \varepsilon_0 \quad \text{and} \quad 0 < \frac{1}{2}\left(\frac{\beta^+}{\beta_{\bar{y}}} - 1\right) < s_0, \quad (3.9)$$

and we set

$$\begin{aligned} \mu &:= \bar{y} - x_0, \\ \lambda &:= \frac{U_\alpha(\bar{y}) - U_\alpha(x_0)}{\beta^+}. \end{aligned} \quad (3.10)$$

Using (3.4), (3.8), and (3.10), we have that

$$0 < \mu < \bar{\delta} \leq 2\alpha/\text{Lip}(F) \quad \text{and} \quad 0 < \lambda \leq \mu < \bar{\delta} \leq \delta. \quad (3.11)$$

Next, we prove that (i) holds. By definition of $\tilde{U}$, (3.11), (3.3) with $y = x_0 - \lambda$, and (3.10), we have that

$$\begin{aligned} \lim_{x \rightarrow (x_0 + \mu)^-} \tilde{U}_\alpha(x) &= U_\alpha(x_0 - \lambda) + (\beta^+ + \beta^-)\lambda \\ &= U_\alpha(x_0) - \beta^- \lambda + (\beta^+ + \beta^-)\lambda = U_\alpha(\bar{y}) = \lim_{x \rightarrow (x_0 + \mu)^+} \tilde{U}_\alpha(x), \end{aligned}$$

from which we conclude that $\tilde{U}_\alpha$ is continuous in $[0, 1]$. Moreover, being absolutely continuous in $[0, 1] \setminus [x_0 - \lambda, x_0 + \mu]$, and piecewise affine in $[x_0 - \lambda, x_0 + \mu]$, we obtain that $\tilde{U}_\alpha \in AC(0, 1)$.

We now show that (ii) is satisfied. Using the admissibility of U_α and the definition of $\tilde{U}$, we are left to prove that

$$F(x) - \alpha \leq U_\alpha(x_0 - \lambda) + \frac{1}{2}(\beta^+ + \beta^-)(x - (x_0 - \lambda)) \leq F(x) + \alpha \quad \text{for all } x \in [x_0 - \lambda, x_0 + \lambda] \quad (3.12)$$

and

$$F(x) - \alpha \leq U_\alpha(x_0 - \lambda) + (\beta^+ + \beta^-)\lambda \leq F(x) + \alpha \quad \text{for all } x \in [x_0 + \lambda, x_0 + \mu]. \quad (3.13)$$

We start by establishing the lower bound in (3.12). Using the inequality $\beta^+ > \beta^-$ and (3.3), we have for all $x \in [x_0 - \lambda, x_0]$ that

$$U_\alpha(x_0 - \lambda) + \frac{1}{2}(\beta^+ + \beta^-)(x - (x_0 - \lambda)) \geq U_\alpha(x_0 - \lambda) + \beta^-(x - (x_0 - \lambda)) = U_\alpha(x) \geq F(x) - \alpha,$$

where we used the admissibility of U_α in the last estimate. On the other hand, the inequality $\beta^+ - \beta^- > 0$ and (3.4) yield for all $x \in [x_0, x_0 + \lambda]$ that

$$\begin{aligned} U_\alpha(x_0 - \lambda) + \frac{1}{2}(\beta^+ + \beta^-)(x - (x_0 - \lambda)) &= U_\alpha(x_0) - \beta^- \lambda + \frac{1}{2}(\beta^+ + \beta^-)(x - (x_0 - \lambda)) \\ &\geq U_\alpha(x) - \beta^+(x - x_0) - \beta^- \lambda + \frac{1}{2}(\beta^+ + \beta^-)(x - (x_0 - \lambda)) = U_\alpha(x) + \frac{1}{2}(\beta^+ - \beta^-)((x_0 + \lambda) - x) \\ &\geq U_\alpha(x) \geq F(x) - \alpha. \end{aligned}$$

Thus, the lower bound in (3.12) holds. To establish the upper bound, we note that for all $x \in [x_0 - \lambda, x_0 + \lambda]$, (3.3) with $y = x_0 - \lambda$, (3.6), (3.8), and (3.11) yield

$$\begin{aligned} U_\alpha(x_0 - \lambda) + \frac{1}{2}(\beta^+ + \beta^-)(x - (x_0 - \lambda)) &= U_\alpha(x_0) - \beta^- \lambda + \frac{1}{2}(\beta^+ + \beta^-)(x - (x_0 - \lambda)) \\ &\leq U_\alpha(x) + |U_\alpha(x_0) - U_\alpha(x)| + \frac{1}{2}|\beta^+ + \beta^-||x - x_0| + \frac{1}{2}|\beta^+ - \beta^-|\lambda \\ &\leq F(x) + \alpha. \end{aligned}$$

Consequently, (3.12) holds.

To prove (3.13), we take $x \in [x_0 + \lambda, x_0 + \mu]$, and observe that the definition of $\tilde{U}_\alpha$, (3.3) with $y = x_0 - \lambda$, (3.10), (3.7), and the admissibility of U_α , used in this order, yield

$$\begin{aligned} \tilde{U}_\alpha(x) &= U_\alpha(x_0 - \lambda) + (\beta^+ + \beta^-)\lambda = U_\alpha(x_0) + \beta^+ \lambda \\ &= U_\alpha(x_0 + \mu) = \max_{[x_0 + \lambda, x_0 + \mu]} U_\alpha \geq U(x) \geq F(x) - \alpha. \end{aligned}$$

Moreover,

$$\tilde{U}_\alpha(x) = U_\alpha(x_0) + \beta^+ \lambda \leq U_\alpha(x) + |U_\alpha(x_0) - U_\alpha(x)| + \beta^+ \lambda \leq F(x) + \alpha,$$

where we used (3.6), (3.8), and (3.11) in the last estimate. Hence, also (3.13) holds.

We are thus left with proving (iii). Using the definition of $\tilde{U}_\alpha$, we need to prove the energy inequality only in the region $[x_0 - \lambda, x_0 + \mu]$; namely,

$$\int_{x_0 - \lambda}^{x_0 + \mu} \sqrt{1 + (\tilde{U}'_\alpha)^2} \, dx < \int_{x_0 - \lambda}^{x_0 + \mu} \sqrt{1 + (U'_\alpha)^2} \, dx. \quad (3.14)$$

Now,

$$\int_{x_0 - \lambda}^{x_0 + \mu} \sqrt{1 + (\tilde{U}'_\alpha)^2} \, dx = (\mu - \lambda) + 2\lambda \sqrt{1 + \frac{1}{4}(\beta^- + \beta^+)^2}, \quad (3.15)$$

and

$$\begin{aligned}
\int_{x_0-\lambda}^{x_0+\mu} \sqrt{1+(U'_\alpha)^2} \, dx &= \lambda \sqrt{1+(\beta^-)^2} + \int_{x_0}^{x_0+\mu} \sqrt{1+(U'_\alpha)^2} \, dx \\
&\geq \lambda \sqrt{1+(\beta^-)^2} + \mu \sqrt{1+\left(\frac{U_\alpha(x_0+\mu)-U_\alpha(x_0)}{\mu}\right)^2} \\
&\geq \lambda \sqrt{1+(\beta^-)^2} + \lambda \sqrt{1+\left(\frac{U_\alpha(x_0+\mu)-U_\alpha(x_0)}{\mu}\right)^2}, \tag{3.16}
\end{aligned}$$

where the previous to last inequality follows from the fact that U_α is affine on $(x_0-\lambda, x_0)$ and concave on $(x_0, x_0+\mu)$, while the last inequality from the fact that $\mu \geq \lambda$. Thus, dividing (3.15) and (3.16) by 2λ , it follows that (3.14) is proved if we show that

$$\frac{\mu-\lambda}{2\lambda} + \sqrt{1+\frac{1}{4}(\beta^-+\beta^+)^2} < \frac{1}{2}\sqrt{1+(\beta^-)^2} + \frac{1}{2}\sqrt{1+\left(\frac{U_\alpha(x_0+\mu)-U_\alpha(x_0)}{\mu}\right)^2}. \tag{3.17}$$

The preceding estimate follows from (3.2) once we obtain that

$$\frac{\mu-\lambda}{2\lambda} < s_0 \quad \text{and} \quad \left| \frac{U_\alpha(x_0+\mu)-U_\alpha(x_0)}{\mu} - \beta^+ \right| < \varepsilon_0. \tag{3.18}$$

By (3.10) and (3.9) (see also (3.5)), we have that

$$\frac{\mu-\lambda}{2\lambda} = \frac{1}{2}(\frac{\beta^+}{\beta_{\bar{y}}} - 1) < s_0 \quad \text{and} \quad \left| \frac{U_\alpha(x_0+\mu)-U_\alpha(x_0)}{\mu} - \beta^+ \right| = |\beta_{\bar{y}} - \beta^+| = \beta^+ - \beta_{\bar{y}} < \varepsilon_0.$$

Hence, (3.18) holds, and so does (3.17) by (3.2). This concludes the proof of (iii), as well of the $\beta^+ > 0$ case.

Finally, we consider the case where $\beta^+ \leq 0$. In this case, the competitor is defined for $x \in [0, 1]$ by

$$\tilde{U}_\alpha(x) := \begin{cases} U_\alpha(x) & \text{if } x < x_0 - \lambda, \\ U_\alpha(x_0 - \lambda) + \frac{1}{2}(\beta^+ + \beta^-)(x - (x_0 - \lambda)) & \text{if } x_0 - \lambda \leq x < x_0 + \lambda, \\ \frac{U_\alpha(x_0 + \mu) - \alpha - U_\alpha(x_0) - \beta\lambda}{\mu - \lambda}(x - x_0 - \lambda) + U_\alpha(x_0) + \beta\lambda & \text{if } x_0 + \lambda \leq x \leq x_0 + \mu, \\ U_\alpha(x) & \text{if } x > x_0 + \mu. \end{cases}$$

In this case, though, we have the freedom to choose λ and μ as we please. We therefore take λ sufficiently small, and $\mu = \lambda + \lambda^2$. Using similar arguments as in the previous case, we conclude.

Step 2. We now consider the case in (DR). The proof follows by applying the argument of Step 1 to the function $V_\alpha(x) := U_\alpha(2x_0 - x)$.

Step 3. We prove case in (UL). The proof follows by applying the argument of Step 1 to the function $V_\alpha(x) := -U_\alpha(x)$.

Step 4. We prove case in (UR). The proof follows by applying the argument of Step 2 to the function $V_\alpha(x) := -U_\alpha(x)$. $\square$

We now prove the monotonicity of the contact sets.

Theorem 3.3. *Under the assumptions of Theorem 3.2, it holds for all $0 < \alpha_1 < \alpha_2 < +\infty$ that*

$$C_{\alpha_1}^+ \cup C_{\alpha_1}^- \supset C_{\alpha_2}^+ \cup C_{\alpha_2}^-. \tag{3.19}$$

Proof. We claim that for every $\bar{\alpha} > 0$, there exists $\beta(\bar{\alpha}) \in (0, \bar{\alpha}]$ such that

$$C_\alpha^+ \cup C_\alpha^- \supset C_{\bar{\alpha}}^+ \cup C_{\bar{\alpha}}^- \quad \text{for all } \alpha \in (\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]. \tag{3.20}$$

Once the preceding claim is proved, we obtain (3.19) by showing that (3.20) implies that

$$\bar{\beta} := \max \left\{ \beta \in (0, \bar{\alpha}]: C_\alpha^+ \cup C_\alpha^- \supset C_{\bar{\alpha}}^+ \cup C_{\bar{\alpha}}^- \text{ for all } \alpha \in (\bar{\alpha} - \beta, \bar{\alpha}] \right\} \tag{3.21}$$

coincides with $\bar{\alpha}$. In fact, assume by contradiction that (3.20) holds but $\bar{\beta} < \bar{\alpha}$. Then, we set $\tilde{\alpha} := \bar{\alpha} - \bar{\beta}$, and use (3.20) to find $\beta(\tilde{\alpha}) \in (0, \tilde{\alpha}]$ such that $C_{\alpha}^+ \cup C_{\alpha}^- \supset C_{\tilde{\alpha}}^+ \cup C_{\tilde{\alpha}}^-$ for all $\alpha \in (\tilde{\alpha} - \beta(\tilde{\alpha}), \tilde{\alpha}] = (\bar{\alpha} - (\bar{\beta} + \beta(\tilde{\alpha})), \bar{\alpha} - \bar{\beta})$. Thus, using (3.21) for $\bar{\beta}$, we conclude that $\gamma := \bar{\beta} + \beta(\tilde{\alpha})$ belongs to the set on the right-hand side of (3.21), which contradicts the maximality of $\bar{\beta}$.

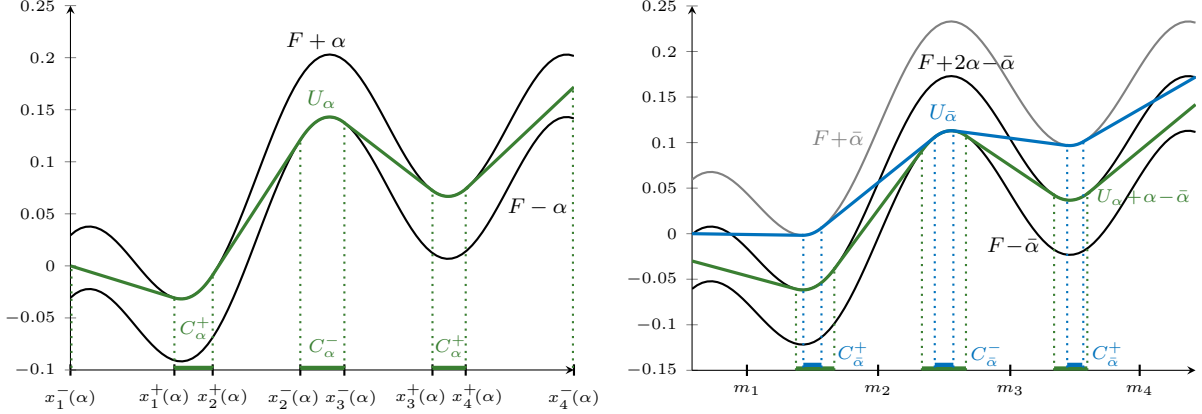


FIGURE 5. The setup of the proof of Theorem 3.3 for f as in Figure 1, $\alpha = 0.03$ and $\bar{\alpha} = 0.06$.

Consequently, we are left to prove the claim in (3.20), illustrated in Figure 5. This will be done in three steps. Before entering into the details, we present the idea of the strategy. First of all, we notice that it is possible to localize the problem, and looking only at intervals of the form $[m_i, m_{i+1}]$ where either $U_{\alpha} < F + \alpha$, or $U_{\alpha} > F - \alpha$. This is done in Step 1. To fix the ideas, let us suppose that $U_{\alpha} < F + \alpha$ in the interval $[m_i, m_{i+1}]$. In order to visualize better what happens, we assume that the lower obstacle is fixed when we move the parameter α . Then, in Step 2 we show that this localized problem solved by U_{α} is equivalent to taking the concave envelope of the function given by the lower obstacle $F - \alpha$ in (m_i, m_{i+1}) and the boundary conditions of U_{α} at the points $\{m_i, m_{i+1}\}$. We now focus on the behavior of the boundary conditions, and we prove in Step 3 that they are monotone with respect to α . This allows us to reach the desired conclusion in Step 4.

Step 1. For each $\alpha > 0$, part 3 of Lemma 2.7 yields that there are only finitely many, say $k(\alpha) \in \mathbb{N}$, pairs of points $x_i^-(\alpha), x_i^+(\alpha) \in [0, 1]$ satisfying for all $i \in \{1, \dots, k(\alpha)\}$ the following four properties:

- (i) $x_i^-(\alpha) \in C_{\alpha}^- \cup \{0, 1\}$, $x_i^+(\alpha) \in C_{\alpha}^+ \cup \{0, 1\}$;
- (ii) U_{α} is affine on the interval having $x_i^-(\alpha)$ and $x_i^+(\alpha)$ as endpoints;
- (iii) The open interval having $x_i^-(\alpha)$ and $x_i^+(\alpha)$ as endpoints does not intersect $C_{\alpha}^- \cup C_{\alpha}^+$;
- (iv) $|x_i^-(\alpha) - x_i^+(\alpha)| \geq \frac{\text{Lip}(F)}{\alpha}$ whenever $x_i^{\pm}(\alpha) \in C_{\alpha}^{\pm}$.

Note that we do not write the explicit interval with $x_i^-(\alpha)$ and $x_i^+(\alpha)$ as endpoints because each of them alternates between being a left endpoint and a right endpoint. Moreover, we observe that not all points in the boundary of the contact set are used, but only those where the solution transitions between the lower and the upper obstacle (see Figure 6), and that the first and the last such points are 0 and 1, respectively.

Fix $\bar{\alpha} > 0$. For each $i = 1, \dots, k(\bar{\alpha})$, let

$$m_i := \frac{x_i^-(\bar{\alpha}) + x_i^+(\bar{\alpha})}{2},$$

and fix

$$0 < \delta < \min \left\{ \frac{|x_i^-(\bar{\alpha}) - x_i^+(\bar{\alpha})|}{2} : i = 1, \dots, k(\bar{\alpha}) \right\}.$$

Note that, by construction, each interval $[m_i, m_{i+1}]$ intersects $C_{\bar{\alpha}}^+$ or $C_{\bar{\alpha}}^-$, but not both. We claim that it is possible to choose $\beta(\bar{\alpha}) > 0$ such that for every $\alpha \in (\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]$ the following hold:

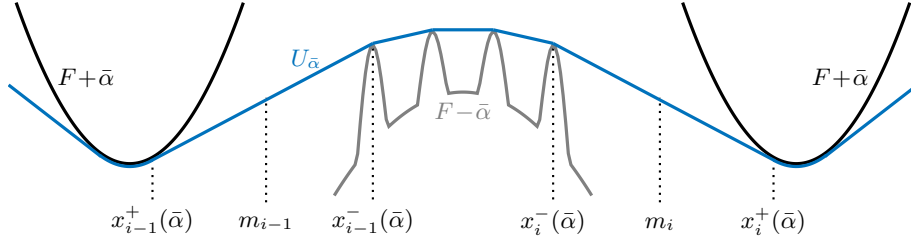


FIGURE 6. Choice of points $x_i^\pm(\bar{\alpha})$ and m_i for the convex envelope argument of Theorem 3.3. In this example, not all points in $\partial C_{\bar{\alpha}}^-$ are used.

- (a) U_α is affine in $(m_i - \delta, m_i + \delta)$;
- (b) Each interval with endpoints m_i and m_{i+1} intersects C_α^+ or C_α^- , but not both;
- (c) There exist $\tilde{\delta}_i^\pm, \bar{\delta}_i^\pm > 0$ such that

$$x_i^-(\alpha) \in (x_i^-(\bar{\alpha}) - \tilde{\delta}_i^-, x_i^-(\bar{\alpha}) + \bar{\delta}_i^-), \quad (3.22)$$

$$x_i^+(\alpha) \in (x_i^+(\bar{\alpha}) - \tilde{\delta}_i^+, x_i^+(\bar{\alpha}) + \bar{\delta}_i^+), \quad (3.23)$$

and

$$(x_i^-(\bar{\alpha}) - \tilde{\delta}_i^-, x_i^-(\bar{\alpha}) + \bar{\delta}_i^-) \cap (x_i^+(\bar{\alpha}) - \tilde{\delta}_i^+, x_i^+(\bar{\alpha}) + \bar{\delta}_i^+) = \emptyset,$$

for all $i \in \{1, \dots, k(\alpha)\}$;

- (d) $k(\alpha) = k(\bar{\alpha})$.

We know that

$$F(x) - \bar{\alpha} < U_{\bar{\alpha}}(x) < F(x) + \bar{\alpha},$$

for every $x \in [m_i - \delta, m_i + \delta]$. By Proposition 2.8, we have that

$$U_\alpha \rightarrow U_{\bar{\alpha}} \quad \text{uniformly as } \alpha \rightarrow \bar{\alpha}. \quad (3.24)$$

Therefore, it is possible to find $\beta(\bar{\alpha}) > 0$ such that

$$F(x) - \alpha < U_\alpha(x) < F(x) + \alpha, \quad (3.25)$$

for every $x \in (m_i - \delta, m_i + \delta)$, and every $\alpha \in (\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]$. By Lemma 2.7, we know that U_α is affine in every connected component of $(0, 1) \setminus (C_\alpha^- \cup C_\alpha^+)$. Thus, by (3.25), we get that (a) is satisfied.

To prove (b), we use the continuity of F and $U_{\bar{\alpha}}$ to fix $\eta > 0$ such that:

- (I) If $F(x) + \bar{\alpha} > U_{\bar{\alpha}}(x)$ for all $x \in [m_i, m_{i+1}]$, then $F(x) + \bar{\alpha} - \eta > U_{\bar{\alpha}}(x)$ for all $x \in [m_i, m_{i+1}]$;
- (II) If $F(x) - \bar{\alpha} < U_{\bar{\alpha}}(x)$ for all $x \in [m_i, m_{i+1}]$, then $F(x) - \bar{\alpha} + \eta < U_{\bar{\alpha}}(x)$ for all $x \in [m_i, m_{i+1}]$.

Note that either (I) or (II) hold because, as mentioned before, each interval $[m_i, m_{i+1}]$ intersects C_α^+ or C_α^- , but not both. Using (3.24), and up to reducing $\beta(\bar{\alpha})$, we get that

$$U_{\bar{\alpha}}(x) - \eta < U_\alpha(x) < U_{\bar{\alpha}}(x) + \eta, \quad \text{for all } x \in [0, 1]. \quad (3.26)$$

Thanks to the choice of η , we obtain (b).

Now, we prove (c). We detail the proof of (3.22), and observe that (3.23) can be established similarly. Consider $x_i^-(\bar{\alpha})$. Just to fix the ideas, assume that

$$(x_i^-(\bar{\alpha}) - \delta, x_i^-(\bar{\alpha})) \subset (0, 1) \setminus (C_{\bar{\alpha}}^- \cup C_{\bar{\alpha}}^+). \quad (3.27)$$

The other case is treated similarly. Up to further reducing the values of $\eta > 0$ and $\beta(\bar{\alpha})$, the continuity of F implies that

$$F(x) - \bar{\alpha} < U_{\bar{\alpha}}(x) - \eta - \beta(\bar{\alpha}), \quad \text{for all } x \in [m_i, x_i^-(\bar{\alpha}) - \delta].$$

Therefore, we get from (3.26) that

$$F(x) - \alpha < F(x) - \bar{\alpha} + \beta(\bar{\alpha}) < U_{\bar{\alpha}}(x) - \eta < U_\alpha(x) \quad \text{for all } x \in [m_i, x_i^-(\bar{\alpha}) - \delta].$$

Thus, setting $\tilde{\delta}_i^- := \delta$,

$$x_i^-(\alpha) > x_i^-(\bar{\alpha}) - \tilde{\delta}_i^-.$$

To prove that there exists $\bar{\delta}_i^- > 0$ satisfying the required property, we consider two cases. First, we treat the situation where

$$U_{\bar{\alpha}}'(x_i^-(\bar{\alpha})) > U_{\bar{\alpha}}'(x_{i+1}^-(\bar{\alpha})). \quad (3.28)$$

Let

$$y_i := \max \left\{ x \geq x_i^-(\bar{\alpha}) : U_{\bar{\alpha}}'(y) = U_{\bar{\alpha}}'(x_i^-(\bar{\alpha})) \text{ for all } y \in [x_i^-(\bar{\alpha}), x] \right\}$$

and

$$z_i := \max \left\{ x \leq x_{i+1}^-(\bar{\alpha}) : U_{\bar{\alpha}}'(y) = U_{\bar{\alpha}}'(x_{i+1}^-(\bar{\alpha})) \text{ for all } y \in [x, x_{i+1}^-(\bar{\alpha})] \right\}.$$

Set

$$\bar{\delta}_i^- := y_i - x_i^-(\bar{\alpha}) + \frac{z_i - y_i}{3}.$$

Up to further reducing $\eta > 0$, thanks to (3.28) and (3.27), we can also assume that

(III) If $F(x) + \bar{\alpha} > U_{\bar{\alpha}}(x)$ for all $x \in [m_i, m_{i+1}]$, let $P_i \in [y_i, z_i]$ be such that

$$U_{\bar{\alpha}}(y_i) + U_{\bar{\alpha}}'(y_i)(P_i - y_i) = U_{\bar{\alpha}}(z_i) + U_{\bar{\alpha}}'(z_i)(P_i - z_i).$$

Then,

$$F(P_i) + \eta < U_{\bar{\alpha}}(y_i) + U_{\bar{\alpha}}'(y_i)(P_i - y_i);$$

(IV) If $F(x) - \bar{\alpha} < U_{\bar{\alpha}}(x)$ for all $x \in [m_i, m_{i+1}]$, then let $Q_i \in [0, 1]$ be such that

$$U_{\bar{\alpha}}(y_i) + U_{\bar{\alpha}}'(y_i)(Q_i - y_i) = U_{\bar{\alpha}}(z_i) + U_{\bar{\alpha}}'(z_i)(Q_i - z_i).$$

Then,

$$F(Q_i) - \eta > U_{\bar{\alpha}}(z_i) + U_{\bar{\alpha}}'(z_i)(Q_i - z_i).$$

Therefore, (c) follows from (3.26), since U_{α} cannot be affine in the entire interval $[x_i^-(\bar{\alpha}), y_i]$.

We now consider the case where (3.28) does not hold. By Lemma 2.7, $U_{\bar{\alpha}}$ is convex in $[x_{i-1}^-(\bar{\alpha}), x_i^-(\bar{\alpha})]$. Therefore, the only other possibility is that

$$U_{\bar{\alpha}}'(x_i^-(\bar{\alpha})) = U_{\bar{\alpha}}'(x_{i+1}^-(\bar{\alpha})).$$

In this case, $U_{\bar{\alpha}}$ is affine in $[x_i^-(\bar{\alpha}), x_{i+1}^-(\bar{\alpha})]$. By Proposition 2.8, Lemma 2.7 (DL), and the choice of η in (I) and (II), only two possibilities can occur: either $x_i^-(\alpha) \in [x_i^-(\bar{\alpha}) - \delta, x_i^-(\bar{\alpha})]$ or $C_{\alpha}^- \cap [x_i^+(\alpha), x_{i+1}^+(\alpha)] = \emptyset$. We aim to exclude the second possibility. Assume, by contradiction, that it holds. Then, a symmetry argument allow us to assume (3.35) without loss of generality. Indeed, if the opposite inequality were true, we could proceed analogously using the function $x \mapsto U_{\alpha}(1 - x)$. At this stage, we choose to refer the reader to Case 2 of Step 3.1 for the final part of the argument, which is identical, instead of repeating it. Our choice is based on the fact that the strategy used in Step 3 forms a central part of the proof of Theorem 3.3, and for the sake of clarity and readability, we prefer to maintain the logical flow of Step 3 without unnecessary interruptions.

Finally, (d) follows directly from (c) and the choice of δ .

Fix $i \in \{1, \dots, k(\alpha)\}$. Now, just to fix the ideas, assume that

$$(m_{i-1}, m_i) \cap C_{\alpha}^+ = \emptyset. \quad (3.29)$$

Then, it is easy to see that $U_{\alpha}|_{[m_{i-1}, m_i]}$ is the unique solution to

$$\min \left\{ \int_0^1 \sqrt{1 + |v'|^2} \, dx : v \in AC(m_{i-1}, m_i), v \geq F - \alpha, v(m_i) = U_{\alpha}(m_{i-1}), v(m_i) = U_{\alpha}(m_{i+1}) \right\}. \quad (3.30)$$

Notice that arguing as in [19, Lemma 2], this minimization problem has a unique solution.

Step 2. We claim that for each $\alpha \in (\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]$, the unique solution to (3.30), and hence $U_{\alpha}|_{[m_{i-1}, m_i]}$, is the concave envelope of the function $G_{\alpha} : [m_{i-1}, m_i] \rightarrow \mathbb{R}$ defined as

$$G_{\alpha}(x) = \begin{cases} U_{\alpha}(m_i) & \text{if } x = m_{i-1}, \\ F(x) - \alpha & \text{if } x \in (m_{i-1}, m_i), \\ U_{\alpha}(m_{i+1}) & \text{if } x = m_i. \end{cases}$$

For clarity, the proof of this fact will be done in two sub-steps.

Step 2.1. First, we show the following. Let $v, w \in AC(m_{i-1}, m_i)$ be such that $v(m_{i-1}) = w(m_{i-1})$, $v(m_i) = w(m_i)$, and $v \leq w$. Assume that v is concave. Then, we claim that

$$\int_{m_{i-1}}^{m_i} \sqrt{1 + |v'|^2} dt \leq \int_{m_{i-1}}^{m_i} \sqrt{1 + |w'|^2} dt.$$

Under the additional assumption that $v \in W^{2,1}(0, 1) \cap C^2(0, 1)$, consider the function $t \mapsto \sqrt{1 + |t|^2}$. For $s, t \in \mathbb{R}$, we have that

$$\begin{aligned} \sqrt{1 + s^2} + \frac{s}{\sqrt{1 + s^2}}(t - s) &= \frac{1 + st}{\sqrt{1 + s^2}} = \frac{(1 + st)\sqrt{1 + t^2}}{\sqrt{(1 + s^2)(1 + t^2)}} \\ &= \frac{\sqrt{(1 + 2st + s^2t^2)(1 + t^2)}}{\sqrt{1 + s^2 + t^2 + s^2t^2}} \leq \sqrt{1 + t^2}, \end{aligned}$$

where we have used $2st \leq s^2 + t^2$. Thus, we get

$$\begin{aligned} \int_{m_{i-1}}^{m_i} \sqrt{1 + |w'|^2} dt &\geq \int_{m_{i-1}}^{m_i} \sqrt{1 + |v'|^2} dt + \int_{m_{i-1}}^{m_i} \frac{v'}{\sqrt{1 + |v'|^2}}(w' - v') dt \\ &= \int_{m_{i-1}}^{m_i} \sqrt{1 + |v'|^2} dt + \left. \frac{v'}{\sqrt{1 + |v'|^2}}(w - v) \right|_{m_{i-1}}^{m_i} \\ &\quad - \int_{m_{i-1}}^{m_i} \frac{v'' + (v'' - 1)(v')^2}{(1 + |v'|^2)^{3/2}}(w - v) dt \\ &= \int_{m_{i-1}}^{m_i} \sqrt{1 + |v'|^2} dt - \int_{m_{i-1}}^{m_i} \frac{v'' + (v'' - 1)(v')^2}{(1 + |v'|^2)^{3/2}}(w - v) dt \\ &\geq \int_{m_{i-1}}^{m_i} \sqrt{1 + |v'|^2} dt, \end{aligned}$$

where the second step follows from integration by parts, and the third by the assumption that v and w have the same boundary conditions. Finally, the inequality is due to the concavity assumption of v , that ensures that $v'' \leq 0$ when it exists, together with $w \geq v$.

To conclude in the general case, we use an approximation argument: let $\{v_n\}_n \in W^{2,1}(m_{i-1}, m_i) \cap C^2(m_{i-1}, m_i)$ and $\{w_n\}_n \in AC(m_{i-1}, m_i) \cap C^2(m_{i-1}, m_i)$ satisfying $v_n(m_i) = w_n(m_i) = v(m_i)$ and $v_n(m_{i-1}) = w_n(m_{i-1}) = v(m_{i-1})$ be such that $v_n \rightarrow v$ strongly in $W^{2,1}$ and $v_n \rightarrow v$ strongly in $W^{1,1}$. Then, by the previous argument, we get that

$$\int_{m_{i-1}}^{m_i} \sqrt{1 + |v'_n|^2} dt \leq \int_{m_{i-1}}^{m_i} \sqrt{1 + |w'_n|^2} dt$$

for all $n \in \mathbb{N}$. Passing to the limit as $n \rightarrow \infty$ gives the desired result.

Step 2.2. We now conclude as follows. First, we note that in the above argument, we could also require that $v \geq G_\alpha$. Indeed, the approximation step can be carried out without warring about this extra constraint.

Therefore, the conclusion of the previous step can be stated as follows: let $v, w \in AC(m_{i-1}, m_i)$ with $v, w \geq F - \alpha$, $v(m_i) = w(m_i) = U_\alpha(m_i)$, $v(m_{i-1}) = w(m_{i-1}) = U_\alpha(m_{i-1})$. Assume that v is concave and that $v \leq w$. Then,

$$\int_{m_{i-1}}^{m_i} \sqrt{1 + |v'|^2} dt \leq \int_{m_{i-1}}^{m_i} \sqrt{1 + |w'|^2} dt.$$

Thus, the solution of the constrained minimization problem (3.30) is given by the smallest concave function that is an admissible competitor. Namely, the concave envelope of G_α .

An alternative proof of Step 2 under more restrictive assumptions can be found in [26].

Step 3. We now present the core of the proof; to be precise, a monotonicity property of the points $U_\alpha(m_i)$, when we consider the graph of $F - \bar{\alpha}$ as a vertical reference. We claim that

$$U_\alpha(m_i) - (\bar{\alpha} - \alpha) \leq U_{\bar{\alpha}}(m_i), \quad (3.31)$$

for all $\alpha \in (\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]$.

To prove (3.31), it is convenient to define the functions

$$\tilde{U}_\alpha := U_\alpha - (\bar{\alpha} - \alpha), \quad \tilde{F}_\alpha := F - (\bar{\alpha} - \alpha), \quad \text{for } \alpha < \bar{\alpha}.$$

Note that $\tilde{U}_\alpha \leq U_\alpha$, $U_\alpha = F \pm \bar{\alpha}$ if and only if $\tilde{U}_\alpha = \tilde{F}_\alpha \pm \bar{\alpha}$, $\tilde{U}_{\bar{\alpha}} = U_{\bar{\alpha}}$, and

$$F - \bar{\alpha} = \tilde{F}_\alpha - \alpha \leq \tilde{U}_\alpha \leq \tilde{F}_\alpha + \alpha = F + 2\alpha - \bar{\alpha}. \quad (3.32)$$

We will prove the validity of (3.31) in two sub-steps.

Step 3.1: We claim that

$$x_i^-(\alpha) \geq x_i^-(\bar{\alpha}), \quad x_i^+(\alpha) \leq x_i^+(\bar{\alpha}), \quad (3.33)$$

for all $\alpha \in (\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]$.

In case $x_i^-(\alpha) \in \{0, 1\}$ or $x_i^+(\alpha) \in \{0, 1\}$, this follows directly from the boundary conditions satisfied by each U_α , for which we always have equality. Next, we only detail the argument to prove that $x_i^-(\alpha) \geq x_i^-(\bar{\alpha})$ for all $\alpha \in (\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]$. The proof of $x_i^+(\alpha) \leq x_i^+(\bar{\alpha})$ follows by arguing similarly.

Note that by (3.29), we have that

$$x_i^-(\bar{\alpha}) < x_i^+(\bar{\alpha}).$$

We start by making two observations. First, we claim that

$$F(x) - \bar{\alpha} \leq U_{\bar{\alpha}}(x_i^-(\bar{\alpha})) + f^-(x_i^-(\bar{\alpha}))(x - x_i^-(\bar{\alpha})), \quad (3.34)$$

for all $x \in (x_i^-(\bar{\alpha}) - \delta, x_i^-(\bar{\alpha}))$, where δ is the one we fixed in Step 1. Indeed, by the choice of δ , we get that $(x_i^-(\bar{\alpha}) - \delta, x_i^-(\bar{\alpha})) \subset (0, 1) \setminus C_{\bar{\alpha}}^-$. Thus, thanks to (1) of Lemma 2.7, we get that $U_{\bar{\alpha}}$ is concave in $x \in (x_i^-(\bar{\alpha}) - \delta, x_i^-(\bar{\alpha}))$. Therefore,

$$F(x) - \bar{\alpha} \leq U_{\bar{\alpha}}(x) \leq U_{\bar{\alpha}}(x_i^-(\bar{\alpha})) + U_{\bar{\alpha}}'(x_i^-(\bar{\alpha}))(x - x_i^-(\bar{\alpha})) \leq U_{\bar{\alpha}}(x_i^-(\bar{\alpha})) + f^-(x_i^-(\bar{\alpha}))(x - x_i^-(\bar{\alpha})),$$

where last inequality follows from (DR) of Theorem 3.2. By contradiction, assume that there exists $\alpha \in (\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]$ such that $x_i^-(\alpha) < x_i^-(\bar{\alpha})$. Then, as we prove next, this implies that

$$U_{\alpha}'^+(x_i^-(\bar{\alpha})) \geq U_{\bar{\alpha}}'^+(x_i^-(\bar{\alpha})). \quad (3.35)$$

Indeed, set

$$y_i^+ := \min\{x_i^+(\alpha), x_i^+(\bar{\alpha})\}, \quad (3.36)$$

and observe that Lemma 2.7 yields that U_α is affine in $[x_i^-(\alpha), y_i^+]$ and $U_{\bar{\alpha}}$ is affine in $[x_i^-(\bar{\alpha}), y_i^+]$. Hence,

$$\frac{U_\alpha(x_i^-(\bar{\alpha})) - U_\alpha(x_i^-(\alpha))}{x_i^-(\bar{\alpha}) - x_i^-(\alpha)} = U_{\alpha}'^-(x_i^-(\bar{\alpha})) = U_{\alpha}'^+(x_i^-(\bar{\alpha})) = \frac{U_\alpha(y_i^+) - U_\alpha(x_i^-(\bar{\alpha}))}{y_i^+ - x_i^-(\alpha)}, \quad (3.37)$$

$$U_{\alpha}'^+(x_i^-(\bar{\alpha})) = \frac{U_{\bar{\alpha}}(y_i^+) - U_{\bar{\alpha}}(x_i^-(\bar{\alpha}))}{y_i^+ - x_i^-(\alpha)}. \quad (3.38)$$

Then,

$$\begin{aligned} U_{\alpha}'^+(x_i^-(\bar{\alpha})) &= \tilde{U}_{\alpha}'^+(x_i^-(\bar{\alpha})) = \frac{\tilde{U}_\alpha(x_i^-(\bar{\alpha})) - \tilde{U}_\alpha(x_i^-(\alpha))}{x_i^-(\bar{\alpha}) - x_i^-(\alpha)} && \text{(by (3.37) and definition of } \tilde{U}_\alpha) \\ &\geq \frac{F(x_i^-(\bar{\alpha})) - \bar{\alpha} - \tilde{U}_\alpha(x_i^-(\alpha))}{x_i^-(\bar{\alpha}) - x_i^-(\alpha)} && \text{(by (3.32))} \\ &= \frac{U_{\bar{\alpha}}(x_i^-(\bar{\alpha})) - (F(x_i^-(\alpha)) - \bar{\alpha})}{x_i^-(\bar{\alpha}) - x_i^-(\alpha)} && \text{(definition of } x_i^-(\bar{\alpha}), x_i^-(\alpha)) \\ &\geq f^-(x_i^-(\bar{\alpha})) && \text{(by (3.34) at } x = x_i^-(\bar{\alpha})) \\ &\geq U_{\bar{\alpha}}'^+(x_i^-(\bar{\alpha})) && \text{(by (DR) of Theorem 3.2),} \end{aligned}$$

which proves (3.35).

Case 1: $x_i^+(\bar{\alpha}) \leq x_i^+(\alpha)$. Recalling (3.36), we have in this case that $y_i^+ = x_i^+(\bar{\alpha})$. Then, using (3.35), (3.37), and (3.38), we deduce that $U_\alpha(x_i^+(\bar{\alpha})) - U_\alpha(x_i^-(\bar{\alpha})) \geq U_{\bar{\alpha}}(x_i^+(\bar{\alpha})) - U_{\bar{\alpha}}(x_i^-(\bar{\alpha}))$ or, equivalently,

$$U_\alpha(x_i^+(\bar{\alpha})) - U_{\bar{\alpha}}(x_i^+(\bar{\alpha})) \geq U_\alpha(x_i^-(\bar{\alpha})) - U_{\bar{\alpha}}(x_i^-(\bar{\alpha})). \quad (3.39)$$

To conclude, we use definition of $x_i^\pm(\bar{\alpha})$ and $x_i^\pm(\alpha)$ and the admissibility conditions of U_α and $U_{\bar{\alpha}}$ to get that

$$U_\alpha(x_i^-(\bar{\alpha})) - U_{\bar{\alpha}}(x_i^-(\bar{\alpha})) \geq F(x_i^-(\bar{\alpha})) - \alpha - (F(x_i^-(\bar{\alpha})) - \bar{\alpha}) = \bar{\alpha} - \alpha,$$

while

$$U_\alpha(x_i^+(\bar{\alpha})) - U_{\bar{\alpha}}(x_i^+(\bar{\alpha})) \leq F(x_i^+(\bar{\alpha})) + \alpha - (F(x_i^+(\bar{\alpha})) + \bar{\alpha}) = \alpha - \bar{\alpha}$$

which contradicts (3.39) because $\alpha < \bar{\alpha}$.

Case 2: $x_i^+(\bar{\alpha}) > x_i^+(\alpha)$. In this case, $y_i^+ = x_i^+(\alpha)$ and $x_i^+(\alpha) \in (x_i^-(\alpha), x_i^+(\bar{\alpha}))$. Because the latter interval is contained in the set $(0, 1) \setminus (C_{\bar{\alpha}}^- \cup C_{\bar{\alpha}}^+)$, it follows that

$$U_{\bar{\alpha}}(x_i^+(\alpha)) < F(x_i^+(\alpha)) + \bar{\alpha}. \quad (3.40)$$

On the other hand, U_α is convex in the interval $[x_i^+(\alpha), x_i^+(\bar{\alpha})]$ by Lemma 2.7; thus,

$$\begin{aligned} U_\alpha(x_i^+(\bar{\alpha})) &\geq U_\alpha(x_i^+(\alpha)) + U_\alpha'(x_i^+(\alpha))(x_i^+(\bar{\alpha}) - x_i^+(\alpha)) \\ &\geq U_\alpha(x_i^+(\alpha)) + U_{\bar{\alpha}}'(x_i^+(\alpha))(x_i^+(\bar{\alpha}) - x_i^+(\alpha)), \end{aligned} \quad (3.41)$$

where, in the last estimate, we have used condition (UL) in Theorem 3.2 and the linearity of U_α in $[x_i^-(\bar{\alpha}), x_i^+(\alpha)]$ to say that

$$U_\alpha'(x_i^+(\alpha)) \geq U_\alpha'(x_i^+(\alpha)) = U_\alpha'(x_i^-(\bar{\alpha})),$$

while (3.35) and the linearity of $U_{\bar{\alpha}}$ in $[x_i^-(\bar{\alpha}), x_i^+(\bar{\alpha})]$ yield

$$U_{\bar{\alpha}}'(x_i^-(\bar{\alpha})) \geq U_{\bar{\alpha}}'(x_i^-(\bar{\alpha})) = U_{\bar{\alpha}}'(x_i^+(\alpha)).$$

Consequently, using the definition of $x_i^+(\bar{\alpha})$ and $x_i^+(\alpha)$ and the admissibility conditions of U_α and $U_{\bar{\alpha}}$, we conclude from (3.40) and (3.41) that

$$\begin{aligned} F(x_i^+(\bar{\alpha})) + \bar{\alpha} &\geq U_\alpha(x_i^+(\bar{\alpha})) - \alpha + \bar{\alpha} \\ &\geq U_\alpha(x_i^+(\alpha)) + U_\alpha'(x_i^+(\alpha))(x_i^+(\bar{\alpha}) - x_i^+(\alpha)) - \alpha + \bar{\alpha} \\ &= F(x_i^+(\alpha)) + \bar{\alpha} + U_\alpha'(x_i^+(\alpha))(x_i^+(\bar{\alpha}) - x_i^+(\alpha)) \\ &> U_{\bar{\alpha}}(x_i^+(\alpha)) + U_{\bar{\alpha}}'(x_i^+(\alpha))(x_i^+(\bar{\alpha}) - x_i^+(\alpha)) \\ &= U_{\bar{\alpha}}(x_i^+(\bar{\alpha})) = F(x_i^+(\bar{\alpha})) + \bar{\alpha}, \end{aligned}$$

where the second to last equality follows from the linearity of $U_{\bar{\alpha}}$ in $[x_i^-(\bar{\alpha}), x_i^+(\bar{\alpha})]$, which yields a contradiction.

This concludes the proof of (3.33).

Step 3.2. We now prove that

$$\tilde{U}_\alpha \leq U_{\bar{\alpha}} \quad \text{in } [x_i^-(\alpha), x_i^+(\alpha)].$$

By Step 3.1, both functions are affine in $[x_i^-(\alpha), x_i^+(\alpha)]$. Thus, it suffices show that

$$\tilde{U}_\alpha(x_i^-(\alpha)) \leq U_{\bar{\alpha}}(x_i^-(\alpha)) \quad (3.42)$$

and

$$\tilde{U}_\alpha(x_i^+(\alpha)) \leq U_{\bar{\alpha}}(x_i^+(\alpha)). \quad (3.43)$$

The estimate (3.42) follows from the fact that

$$\tilde{U}_\alpha(x_i^-(\alpha)) = \tilde{F}(x_i^-(\alpha)) - \alpha = F(x_i^-(\alpha)) - \bar{\alpha} \leq U_{\bar{\alpha}}(x_i^-(\alpha)).$$

To prove (3.43), we argue as follows. In view of (3.33), we can reverse the roles of U_α and $U_{\bar{\alpha}}$ in (3.35) to get

$$U_{\bar{\alpha}}'(x_i^-(\alpha)) \geq U_{\bar{\alpha}}'(x_i^-(\bar{\alpha})) = \tilde{U}_\alpha'(x_i^-(\bar{\alpha})),$$

which, by linearity, implies that $U_{\bar{\alpha}}(x_i^+(\alpha)) - U_{\bar{\alpha}}(x_i^-(\alpha)) \geq \tilde{U}_\alpha(x_i^+(\alpha)) - \tilde{U}_\alpha(x_i^-(\alpha))$. Thus,

$$U_{\bar{\alpha}}(x_i^+(\alpha)) - \tilde{U}_\alpha(x_i^+(\alpha)) \geq U_{\bar{\alpha}}(x_i^-(\alpha)) - \tilde{U}_\alpha(x_i^-(\alpha)) \geq 0,$$

where we used (3.42) in the last estimate. Hence, (3.43) also holds.

If instead of (3.29), i is such that

$$(m_{i-1}, m_i) \cap C_\alpha^- = \emptyset, \quad (3.44)$$

by using the same symmetry argument as in Theorem 3.2, we conclude the validity of (3.31) also in this case. In particular, we obtain that

$$U_\alpha(m_i) - (\bar{\alpha} - \alpha) \leq U_{\bar{\alpha}}(m_i), \quad (3.45)$$

for all $i \in \{1, \dots, k(\bar{\alpha})\}$, and for all $\alpha \in [\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]$.

Step 4. We now conclude as follows. To fix the ideas, let's assume that (3.29) is in force. For each $\alpha \in (\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]$, Step 2 yields that $\tilde{U}_\alpha|_{[m_{i-1}, m_i]}$ is the concave envelope of the function $\tilde{G}_\alpha : [m_{i-1}, m_i] \rightarrow \mathbb{R}$ defined by

$$\tilde{G}_\alpha(x) := G_\alpha - (\bar{\alpha} - \alpha) = \begin{cases} \tilde{U}_\alpha(m_{i-1}) & \text{if } x = m_{i-1}, \\ F(x) - \bar{\alpha} & \text{if } x \in (m_{i-1}, m_i), \\ \tilde{U}_\alpha(m_i) & \text{if } x = m_i. \end{cases} \quad (3.46)$$

On the other hand, by (3.45), we have for all $\alpha \in [\bar{\alpha} - \beta(\bar{\alpha}), \bar{\alpha}]$ that

$$\tilde{U}_\alpha(m_{i-1}) \leq U_{\bar{\alpha}}(m_{i-1}), \quad \tilde{U}_\alpha(m_i) \leq U_{\bar{\alpha}}(m_i).$$

Thus,

$$\tilde{G}_\alpha \leq G_{\bar{\alpha}},$$

which implies that $\tilde{U}_\alpha \leq U_{\bar{\alpha}}$ by taking the concave envelope in the preceding estimate. Consequently,

$$\tilde{G}_\alpha \leq \tilde{U}_\alpha \leq U_{\bar{\alpha}}. \quad (3.47)$$

Hence, if $x \in C_{\bar{\alpha}}^-$, we get from (3.46) that

$$U_{\bar{\alpha}}(x) = F(x) - \bar{\alpha} = G_{\bar{\alpha}}(x) = \tilde{G}_\alpha(x).$$

Therefore, using (3.47), we obtain that $\tilde{U}_\alpha(x) = F(x) - \bar{\alpha}$. Using the definition of $\tilde{U}_\alpha$, this is equivalent to saying that $x \in C_{\bar{\alpha}}^-$. Hence, $C_{\bar{\alpha}}^- \supset C_{\bar{\alpha}}^-$. If instead of (3.29), we have (3.44), then by using a symmetry argument, we get that $C_{\bar{\alpha}}^+ \supset C_{\bar{\alpha}}^+$. $\square$

We now investigate the relation between the jump set of u_α and that of f .

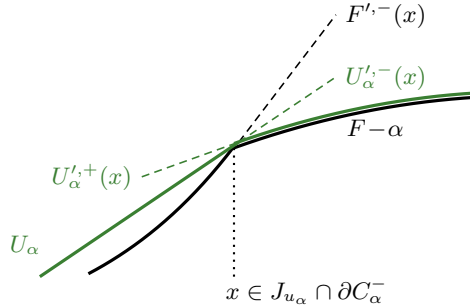


FIGURE 7. Jumps of $u_\alpha = U'_\alpha$ must be jumps of $f = F'$, even if they happen at the boundary of the contact set, cf. (3.50) in Proposition 3.4.

Proposition 3.4. *Let $f \in L^\infty(0, 1)$ be such that the approximate left and right limits in (1.3) exist for every $x_0 \in (0, 1)$. Then, for every $\alpha \in (0, +\infty)$, we have that*

$$J_{u_\alpha} \subset J_f \cap (C_\alpha^+ \cup C_\alpha^-), \quad (3.48)$$

where J_f is (by definition) the set of points where both limits in (1.3) exist but are different. More precisely, we have

$$J_{u_\alpha} \cap \text{int}(C_\alpha^+ \cup C_\alpha^-) = J_f \cap \text{int}(C_\alpha^+ \cup C_\alpha^-), \quad (3.49)$$

as well as (see Figure 7)

$$J_{u_\alpha} \cap \partial(C_\alpha^+ \cup C_\alpha^-) \subset J_f. \quad (3.50)$$

Proof. We first prove (3.49). We observe that, since $x_0 \in \text{int } C_\alpha^\pm$, it is possible to find $\delta_{x_0} > 0$ such that $I_{x_0} := (x_0 - \delta_{x_0}, x_0 + \delta_{x_0}) \subset \text{int } C_\alpha^\pm$; in other words, using the definition of $C_\alpha^\pm$, such that

$$U_\alpha(x) = F(x) \pm \alpha \quad \text{for all } x \in I_{x_0}.$$

Thus, recalling Remark 3.1 and using the definition of F in (2.1), we conclude that

$$U'_\alpha = f \quad \mathcal{L}^1\text{-a.e. in } I_{x_0}.$$

Consequently, because $x_0 \in \text{int } C_\alpha^\pm$ was taken arbitrarily, we conclude that (3.49) holds, with $f \in BV(\text{int}(C_\alpha^+ \cup C_\alpha^-))$ by Theorem 2.2.

We now prove (3.50). Let us assume for a contradiction that there is some $x_0 \in (0, 1)$ with

$$x_0 \in (J_{u_\alpha} \cap \partial(C_\alpha^+ \cup C_\alpha^-)) \setminus J_f.$$

Without loss of generality, we can assume that $x_0 \in C_\alpha^-$, and that there exists $\delta > 0$ such that $U(x) > F(x) - \alpha$ for all $x \in (x_0 - \delta, x_0)$. The proof in the other cases uses a similar strategy. Thanks to case (DL) of Theorem 3.2, we have that

$$U_{\alpha'}^{'+}(x_0) < U_{\alpha'}^{'-}(x_0), \quad (3.51)$$

where the strict inequality is due to the assumption that $x_0 \in J_{u_\alpha}$. Now, since $U \geq F - \alpha$, and $F(x_0) - \alpha = U_\alpha(x_0)$, we get that

$$\frac{F(x) - F(x_0)}{x - x_0} \geq \frac{U_\alpha(x) - U_\alpha(x_0)}{x - x_0},$$

for $x < x_0$, and

$$\frac{F(x) - F(x_0)}{x - x_0} \leq \frac{U_\alpha(x) - U_\alpha(x_0)}{x - x_0}$$

for $x > x_0$. Passing to the limit as $x \rightarrow x_0^-$ and $x \rightarrow x_0^+$, respectively, and using the absurd assumption that $x \notin J_f$, we get

$$U_{\alpha'}^{'+}(x_0) \geq f(x_0) \geq U_{\alpha'}^{'-}(x_0),$$

which is the desired contradiction with (3.51).

Finally, we show the validity of (3.48). By Theorem 2.2, we have that

$$J_{u_\alpha} = J_{U'_\alpha}.$$

By Lemma 2.7, U'_α is constant on each connected component of the open set $(0, 1) \setminus (C_\alpha^- \cup C_\alpha^+)$. Thus,

$$J_{U'_\alpha} \subset (0, 1) \setminus ((0, 1) \setminus (C_\alpha^- \cup C_\alpha^+)) = C_\alpha^+ \cup C_\alpha^-.$$

Using (3.49) and (3.50), we conclude the proof. $\square$

Remark 3.5. In the case of ROF on higher dimensional domains, the inclusion (3.48) can be proved by several different methods (either comparison of level sets seen as generalized surfaces of prescribed mean curvature as in [7] or different flavors of inner variations [32, 11]), but in those cases one needs to assume $f \in L^\infty \cap BV$.

Remark 3.6. We emphasize that the opposite inclusion to (3.50) does not hold, as can be seen by taking $f = \mathbf{1}_{(1/2, 1)}$ and $\alpha > 1/2$, in which case $u \equiv 1/2$.

We finally conclude the proof by proving that the monotonicity of the contact set implies the monotonicity of the jump set.

Proposition 3.7. *Let $\alpha_1, \alpha_2 \in (0, +\infty)$ with $\alpha_1 < \alpha_2$, and assume that*

$$C_{\alpha_1}^+ \cup C_{\alpha_1}^- \supset C_{\alpha_2}^+ \cup C_{\alpha_2}^-.$$

Then,

$$J_{u_{\alpha_1}} \supset J_{u_{\alpha_2}}.$$

Proof. Let $x \in J_{u_{\alpha_2}}$. By (3.48) of Proposition 3.4, we get that $x \in J_f \cap (C_{\alpha_2}^+ \cup C_{\alpha_2}^-)$. Thus, by our assumption, we infer that $x \in J_f \cap (C_{\alpha_1}^+ \cup C_{\alpha_1}^-)$. First, consider the case where $x \in J_f \cap \text{int}(C_{\alpha_1}^+ \cup C_{\alpha_1}^-)$. By using (3.49), we obtain that $x \in J_{u_{\alpha_1}}$.

Now, consider the case where $x \in J_f \cap \partial(C_{\alpha_1}^+ \cup C_{\alpha_1}^-)$. Assume $x \in J_f \cap \partial C_{\alpha_1}^-$. In the case $x \in J_f \cap \partial C_{\alpha_1}^+$, the conclusion follows by using a similar argument. We argue by contradiction, namely we assume that $x \notin J_{u_{\alpha_1}}$. This yields that

$$U_{\alpha_1}^{\prime,+}(x) = U_{\alpha_1}^{\prime,-}(x).$$

We claim that one the following options holds:

$$U_{\alpha_1}^{\prime,+}(x) > U_{\alpha_2}^{\prime,+}(x), \quad (3.52)$$

or

$$U_{\alpha_1}^{\prime,-}(x) > U_{\alpha_2}^{\prime,-}(x). \quad (3.53)$$

Indeed, note that we must have

$$U_{\alpha_2}^{\prime,+}(x) < U_{\alpha_2}^{\prime,-}(x), \quad (3.54)$$

where the strict inequality is due to the fact that $x \in J_{u_{\alpha_2}}$. Thus, if (3.53) does not hold, then $U_{\alpha_2}^{\prime,-}(x) \leq U_{\alpha_1}^{\prime,-}(x)$. Therefore, from (3.54) we conclude that (3.52) is in force.

Thus, assume we are in the case where (3.52) holds. The case where (3.53) holds is treated with similar arguments. We claim that there exists $x_{\alpha_1} > x$ such that U_{α_1} is affine in $[x, x_{\alpha_1}]$. Indeed, since

$$U_{\alpha_1}^{\prime,+}(x) > U_{\alpha_2}^{\prime,+}(x), \quad U_{\alpha_1}(x) = F(x) - \alpha_1, \quad U_{\alpha_2}(x) = F(x) - \alpha_2,$$

we obtain that there exists $x_0 > x$ such that

$$U_{\alpha_1}(y) - (\alpha_2 - \alpha_1) > U_{\alpha_2}(y), \quad (3.55)$$

for all $y \in (x, x_0)$. Since $\alpha_1 > 0$, there exists $\delta > 0$ such that $U_{\alpha_1}(y) \notin C_{\alpha_1}^+$ for $y \in [x, x + \delta]$. This proves the claim.

We can thus assume that $x_{\alpha_1} > x$ is the largest value for which U_{α_1} is affine in $[x, x_{\alpha_1}]$. We then have three possible cases.

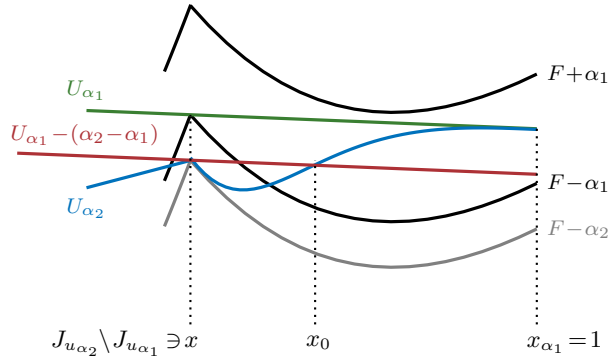


FIGURE 8. Case 1 in the proof of Proposition 3.7, a competitor is built by replacing U_{α_2} by $U_{\alpha_1} - (\alpha_2 - \alpha_1)$ on $[x, x_0]$.

Case 1: $x_{\alpha_1} = 1$. Using (3.55), together with the fact that $U_{\alpha_1}(1) = F(1) = U_{\alpha_2}(1)$, we obtain that there exists $x_0 \in (x, 1]$ such that

$$U_{\alpha_1}(x_0) - (\alpha_2 - \alpha_1) = U_{\alpha_2}(x_0),$$

and

$$F(y) - \alpha_2 \leq U_{\alpha_2}(y) < U_{\alpha_1}(y) - (\alpha_2 - \alpha_1),$$

for all $y \in [x, x_0]$. Note that, by (3.52) and the above, U_{α_2} cannot be affine in the entire interval $[x, x_0]$. But this then contradicts the minimality of U_{α_2} , since in that case the graph of the function

$$v(y) := \begin{cases} U_{\alpha_1}(y) - (\alpha_2 - \alpha_1) & \text{if } y \in [x, x_0], \\ U_{\alpha_2}(y) & \text{else,} \end{cases}$$

has strictly less length than that of U_{α_2} , as illustrated in Figure 8. Note that the function v is such that $v(0) = F(0)$, $v(1) = F(1)$, and $\|v - F\|_\infty \leq \alpha_2$. This gives the desired contradiction.

Case 2: $x_{\alpha_1} \in C_{\alpha_1}^-$. We claim that there exists $x_0 \in (x, 1]$ such that

$$U_{\alpha_1}(x_0) - (\alpha_2 - \alpha_1) = U_{\alpha_2}(x_0),$$

and

$$F(y) - \alpha_2 \leq U_{\alpha_2}(y) < U_{\alpha_1}(y) - (\alpha_2 - \alpha_1),$$

for all $y \in [x, x_0]$. This follows from the continuity of F , U_{α_1} , and U_{α_2} together with the fact that $U_{\alpha_1}(x_{\alpha_1}) - (\alpha_2 - \alpha_1) = F(x_{\alpha_1}) - \alpha_2$. We then conclude with a similar argument to that employed in Case 1.

Case 3: $x_{\alpha_1} \in C_{\alpha_1}^+$. Let $\bar{\alpha} > 0$ be such that

$$\|U_\alpha - U_{\alpha_1}\|_\infty < \frac{\alpha_1}{2}, \quad (3.56)$$

for all $\alpha \in (\alpha_1, \bar{\alpha})$. Take $\alpha \in (\alpha_1, \bar{\alpha})$. Thanks to Proposition 2.8, we know that there exist $\delta, \mu \in \mathbb{R}$ such that U_α is affine in $[x + \delta, x_{\alpha_1} + \mu]$. We claim that $\mu > 0$. Indeed, since $F + \alpha > F + \alpha_1$, we obtain that U_α cannot touch the graph of $F + \alpha$ in the interval $[x + \delta, x_{\alpha_1} + \mu]$. Moreover, thanks to (3.56), we have that $U_\alpha > F - \alpha$ in the same interval. This proves the claim.

Let $x_\alpha > x$ be the largest value for which U_α is affine in $[x, x_\alpha]$. we now have three cases to consider.

Case 3.1: $x_\alpha \in C_\alpha^+$. With a similar argument to that used in case 2, with the role of U_{α_1} played by U_α , and that of U_{α_2} played by U_{α_1} , we conclude.

Case 3.2: $x_\alpha = 1$. In this case we use an argument similar to those of case 1.

Case 3.3: $x_\alpha \in C_\alpha^-$. We claim that, up to taking α sufficiently close to α_1 , this is not possible. Let's argue by contradiction. Assume there exists $(\alpha_n)_{n \in \mathbb{N}}$ with $\alpha_n > \alpha_1$ and $\alpha_n \rightarrow \alpha$ when $n \rightarrow \infty$ such that x_{α_n} , the largest value where U_{α_n} is affine in $[x, x_{\alpha_n}]$, is such that $x_{\alpha_n} \in C_{\alpha_n}^-$. Then, up to extracting a subsequence, we would have $x_{\alpha_n} \rightarrow x_0$. Since $U_{\alpha_n} \rightarrow U_\alpha$ uniformly, we must have $x_0 = x_{\alpha_1}$. Therefore, using again the uniform convergence, we would get

$$\lim_{n \rightarrow \infty} U_{\alpha_n}(x_n) = U_{\alpha_1}(x_{\alpha_1}).$$

This is not possible, since

$$U_{\alpha_n}(x_n) = F(\alpha_n) - \alpha_n > F(\alpha_n) + \alpha_n \rightarrow F(x_{\alpha_1}) + \alpha_1 = U_{\alpha_1} + \alpha_1.$$

Since $\alpha_1 > 0$, the strict inequality above does not saturate. This gives the desired contradiction and concludes the proof. $\square$

Finally, we prove the monotonicity of the amplitude of the jump.

Proposition 3.8. *Let $\alpha_1 < \alpha_2$. Then, it holds that*

$$|u_{\alpha_1}^l(x) - u_{\alpha_1}^r(x)| \geq |u_{\alpha_2}^l(x) - u_{\alpha_2}^r(x)|,$$

for all $x \in J_{u_{\alpha_2}}$.

Proof. Note that $x \in J_f$ if and only if x is not a point of differentiability for F . In particular, thanks to our assumption, if $F'^-, (x) \neq F'^+, (x)$, namely if the left and the right slopes of F differ.

Fix $\alpha_1 < \alpha_2$, and let $x \in J_{u_{\alpha_2}}$. By Proposition 3.4, we have two possibilities:

$$x \in J_f \cap \text{int}(C_{\alpha_2}^+ \cup C_{\alpha_2}^-), \quad \text{or} \quad x \in J_f \cap \partial(C_{\alpha_2}^+ \cup C_{\alpha_2}^-).$$

In the first case, assume without loss of generality that $x \in J_f \cap \text{int} C_{\alpha_2}^+$. Then, there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset \text{int} C_{\alpha_2}^+$. This implies that

$$U_{\alpha_2}'^-(x) = F'^-, (x), \quad U_{\alpha_2}'^+(x) = F'^+, (x).$$

The monotonicity of the contact set with respect of α (see Theorem 3.3) implies that $x \in J_f \cap \text{int} C_{\alpha_1}^+$. Therefore,

$$U_{\alpha_2}'^-(x) = U_{\alpha_1}'^-(x), \quad U_{\alpha_2}'^+(x) = U_{\alpha_1}'^+(x),$$

and thanks to Theorem 2.2, we get that $u_{\alpha_2}^l(x) = u_{\alpha_1}^l(x)$ and $u_{\alpha_2}^r(x) = u_{\alpha_1}^r(x)$.

In the second case, assume without loss of generality that $x \in J_f \cap \partial C_{\alpha_2}^-$. We consider two sub-cases:

$$x \in J_f \cap \text{int } C_{\alpha_1}^- \quad \text{or} \quad x \in J_f \cap \partial C_{\alpha_1}^-.$$

In both of these sub-cases, we assume without loss of generality that there exists $\delta > 0$ such that $(x - \delta, x) \subset (0, 1) \setminus (C_{\alpha_2}^+ \cup C_{\alpha_2}^-)$.

In the first sub-case, note that

$$U_{\alpha_1}'^+(x) = f^+(x), \quad f^-(x) = U_{\alpha_1}'^-(x)$$

Then, (DL) of Theorem 3.2 yields that

$$U_{\alpha_1}'^+(x) = f^+(x) \leq U_{\alpha_2}'^+(x) \leq U_{\alpha_2}'^-(x) \leq f^-(x) = U_{\alpha_1}'^-(x).$$

Using Theorem 2.2, we obtain the desired inequality.

In the second sub-case, we claim that

$$U_{\alpha_2}'^-(x) \leq U_{\alpha_1}'^-(x),$$

and that

$$U_{\alpha_1}'^+(x) \leq U_{\alpha_2}'^+(x).$$

Let us start by proving the first claim. Thanks to Theorem 3.3, up to reducing $\delta > 0$, we can assume that $(x - \delta, x) \subset (0, 1) \setminus (C_{\alpha_1}^+ \cup C_{\alpha_1}^-)$. Recall that in Step 3 of the proof of Theorem 3.3, we proved that

$$U_{\alpha_1}(x - \delta) - (\alpha_2 - \alpha_1) \leq U_{\alpha_2}(x - \delta).$$

Therefore, since U_{α_1} and U_{α_2} are affine in $(x - \delta, x)$, and $U_{\alpha_1}(x) = U_{\alpha_2}(x)$, we obtain the desired result.

To prove the second claim, we notice that if we are able to reduce δ so that $(x, x + \delta) \subset C_{\alpha_1}^-$, then we conclude by using (DL) of Theorem 3.2. Otherwise, we argue as in the proof of the previous claim. $\square$

ACKNOWLEDGEMENTS

The research of Riccardo Cristoferi was partially supported under NWO-OCENW.M.21.336. Rita Ferreira was partially supported by King Abdullah University of Science and Technology (KAUST) baseline funds and KAUST OSR-CRG2021-4674. The research of Irene Fonseca was partially supported by the National Science Foundation under grants DMS-2108784, DMS-2205627, and DMS-2342349. Part of this work was carried out while Riccardo Cristoferi and José A. Iglesias attended the Workshop Calculus of Variations NL 2024 in Schiermonnikoog, for which those authors are grateful. All authors also thank the hospitality of the Erwin Schrödinger International Institute for Mathematics and Physics (ESI) in Vienna, where part of this work was carried out.

REFERENCES

- [1] R. Acar and C. R. Vogel. Analysis of bounded variation penalty methods for ill-posed problems. *Inverse Problems*, 10(6):1217–1229, 1994.
- [2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. Oxford University Press, New York, 2000.
- [3] U. Bauer, A. Munk, H. Sieling, and M. Wardetzky. Persistence barcodes versus Kolmogorov signatures: detecting modes of one-dimensional signals. *Found. Comput. Math.*, 17(1):1–33, 2017.
- [4] K. Bredies, J. A. Iglesias, and G. Mercier. Boundedness and unboundedness in total variation regularization. *Appl. Math. Optim.*, 88(2):Paper No. 51, 42, 2023.
- [5] K. Bredies, K. Kunisch, and T. Pock. Total generalized variation. *SIAM Journal on Imaging Sciences*, 3(3):492–526, 2010.
- [6] A. Briani, A. Chambolle, M. Novaga, and G. Orlandi. On the gradient flow of a one-homogeneous functional. *Confluentes Math.*, 3(4):617–635, 2011.
- [7] V. Caselles, A. Chambolle, and M. Novaga. The discontinuity set of solutions of the TV denoising problem and some extensions. *Multiscale Model. Simul.*, 6(3):879–894, 2007.
- [8] V. Caselles, K. Jalalzai, and M. Novaga. On the jump set of solutions of the total variation flow. *Rend. Semin. Mat. Univ. Padova*, 130:155–168, 2013.
- [9] A. Chambolle. An algorithm for total variation minimization and applications. *Journal of Mathematical Imaging and Vision*, 20(1):89–97, 2004.
- [10] A. Chambolle, V. Duval, G. Peyré, and C. Poon. Geometric properties of solutions to the total variation denoising problem. *Inverse Problems*, 33(1):015002, 44, 2017.

- [11] A. Chambolle and M. Lasica. Inclusion and estimates for the jumps of minimizers in variational denoising. *SIAM J. Imaging Sci.*, 17(3):1844–1878, 2024.
- [12] A. Chambolle and T. Pock. A first-order primal-dual algorithm for convex problems with applications to imaging. *J. Math. Imaging Vision*, 40(1):120–145, 2011.
- [13] L. Condat. A direct algorithm for 1-D total variation denoising. *IEEE Signal Process Lett.*, 20(11):1054–1057, 2013.
- [14] R. Cristoferi. Exact solutions for the total variation denoising problem of piecewise constant images in dimension one. *J. Appl. Anal.*, 27(1):13–33, 2021.
- [15] G. Dal Maso. *An introduction to Γ -convergence*, volume 8 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1993.
- [16] P. L. Davies and A. Kovac. Local extremes, runs, strings and multiresolution. *Ann. Statist.*, 29(1):1–65, 2001. With discussion and rejoinder by the authors.
- [17] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.
- [18] L. Giacomelli and M. Lasica. A local estimate for vectorial total variation minimization in one dimension. *Nonlinear Anal.*, 181:141–146, 2019.
- [19] M. Grasmair. The equivalence of the taut string algorithm and BV-regularization. *J. Math. Imaging Vision*, 27(1):59–66, 2007.
- [20] M. Grasmair and F. Lenzen. Anisotropic total variation filtering. *Appl. Math. Optim.*, 62(3):323–339, 2010.
- [21] M. Grasmair and A. Obereder. Generalizations of the taut string method. *Numer. Funct. Anal. Optim.*, 29(3-4):346–361, 2008.
- [22] W. Hinterberger, M. Hintermüller, K. Kunisch, M. von Oehsen, and O. Scherzer. Tube methods for BV regularization. *J. Math. Imaging Vision*, 19(3):219–235, 2003.
- [23] J. A. Iglesias, G. Mercier, and O. Scherzer. A note on convergence of solutions of total variation regularized linear inverse problems. *Inverse Problems*, 34(5):055011, 28, 2018.
- [24] K. Jalalzai. *Regularization of inverse problems in image processing*. PhD thesis, Ecole Polytechnique X, 2012.
- [25] K. Jalalzai and A. Chambolle. Properties of minimizers of the total variation and of the solutions of the total variation flow. In *2014 IEEE International Conference on Image Processing (ICIP)*, pages 4832–4836, 2014.
- [26] F. Kadhi and A. Trad. Characterization and approximation of the convex envelope of a function. *J. Optim. Theory Appl.*, 110(2):457–466, 2001.
- [27] M. Lasica, S. Moll, and P. B. Mucha. Total variation denoising in l^1 anisotropy. *SIAM J. Imaging Sci.*, 10(4):1691–1723, 2017.
- [28] G. Leoni. *A first course in Sobolev spaces*, volume 181 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2017.
- [29] N. C. Overgaard. On the taut string interpretation and other properties of the Rudin-Osher-Fatemi model in one dimension. *J. Math. Imaging Vision*, 61(9):1276–1300, 2019.
- [30] L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Phys. D*, 60(1-4):259–268, 1992.
- [31] O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier, and F. Lenzen. *Variational methods in imaging*, volume 167 of *Applied Mathematical Sciences*. Springer, New York, 2009.
- [32] T. Valkonen. The jump set under geometric regularization. Part 1: basic technique and first-order denoising. *SIAM J. Math. Anal.*, 47(4):2587–2629, 2015.

RADBOUD UNIVERSITY, IMAPP - DEPARTMENT OF MATHEMATICS, PO BOX 9010, 6500 GL NIJMEGEN, THE NETHERLANDS

Email address: `cristoferi@science.ru.nl`

KING ABDULLAH UNIVERSITY OF SCIENCE AND TECHNOLOGY (KAUST), CEMSE DIVISION, THUWAL 23955-6900, SAUDI ARABIA

Email address: `rita.ferreira@kaust.edu.sa`

CARNEGIE MELLON UNIVERSITY, 5000 FORBES AVENUE, PITTSBURGH, PA 15213

Email address: `fonseca@andrew.cmu.edu`

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF TWENTE, P.O. BOX 217, 7500 AE ENSCHEDE, THE NETHERLANDS

Email address: `jose.iglesias@utwente.nl`